# On the characterization of a family of generalized Yager's implications

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# Outline

1. The unavoidable characterization of families of fuzzy implication functions
   - Fuzzy implication functions
   - Additional Properties
   - The need of characterization
2. From $(h, e)$-implications to some generalizations of Yager's implications
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# The unavoidable characterization of families of fuzzy implication functions

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- more than 40 fuzzy implication functions have been used in control theory,
- a large number of families of fuzzy implication functions have been proposed.

# Fuzzy implication functions

The definition of fuzzy implication function is enough flexible to allow the existence of a huge number of fuzzy implication functions.

**Definition**

A binary operation $I : [0,1]^2 \to [0,1]$ is said to be a *fuzzy implication function* if it satisfies:

(I1) $I(x,z) \geq I(y,z)$ when $x \leq y$, for all $z \in [0,1]$.

(I2) $I(x,y) \leq I(x,z)$ when $y \leq z$, for all $x \in [0,1]$.

(I3) $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.

# Additional properties

These operators can satisfy additional properties that come usually from tautologies in classical logic.

1. *Exchange Principle*:

$$I(x, I(y, z)) = I(y, I(x, z)), \quad \text{for all } x, y, z \in [0, 1]. \qquad \textbf{(EP)}$$

2. *Law of importation* with respect to a t-norm $T$:

$$I(T(x, y), z) = I(x, I(y, z)), \quad \text{for all } \quad x, y, z \in [0, 1]. \qquad (\mathbf{LI_T})$$

3. *Left-neutrality principle*:

$$I(1, y) = y \qquad \text{for all } \quad y \in [0, 1]. \qquad \textbf{(NP)}$$

4. The *left neutrality principle with e* $\in ]0, 1[$,

$$I(e, y) = y, \quad y \in [0, 1]. \qquad (\mathbf{NP_e})$$

# Strategies to generate new families

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2. Those based on the use of additive univalued generators: Yager's $f$ and $g$-generated implications, $h$ and $(h, e)$-implications, etc.

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1. Those based on combinations of other logical connectives (aggregation functions, fuzzy negations…): $(S, N)$, $R$, $QL$, $D$, etc.
2. Those based on the use of additive univalued generators: Yager's $f$ and $g$-generated implications, $h$ and $(h, e)$-implications, etc.
3. Those based on the use of other fuzzy implication functions: $\varphi$-conjugation, threshold horizontal and vertical methods, etc.

# The need of characterization

From time to time, “new” families of fuzzy implication functions appear. However, some time later some of them are proved to have intersection with other old families or even they are actually the same family!

**Solution**

To axiomatically characterize the families of fuzzy implication functions in order to know better their structure and behaviour.

# $(h, e)$-implications

There are still some families of fuzzy implication functions which have not been characterized yet. One of them is the family of $(h, e)$-implications.

**Definition (Massanet, Torrens (2011))**

Let $h : [0, 1] \to [-\infty, \infty]$ be a strictly increasing and continuous function with $h(0) = -\infty$, $h(e) = 0$ for an $e \in (0, 1)$ and $h(1) = +\infty$. The function $I : [0, 1]^2 \to [0, 1]$ defined by

$$I^{h,e}(x, y) = \begin{cases} 1 & \text{if } x = 0, \\ h^{-1}\left(\frac{x}{e} \cdot h(y)\right) & \text{if } x > 0 \text{ and } y \leq e, \\ h^{-1}\left(\frac{e}{x} \cdot h(y)\right) & \text{if } x > 0 \text{ and } y > e, \end{cases}$$

is called an $(h, e)$-implication. The function $h$ is called an $h$-generator of $I_{h,e}$.

# Importance of $(h, e)$-implications

They satisfy a controlled increasingness in the second variable.

**Theorem (Massanet, Torrens (2012))**

*Let $h$ be an $h$-generator with respect to a fixed $e \in (0, 1)$. Then the following properties hold:*

(i) *If $x > 0$ and $y < e$, then $I^{h,e}(x, y) < e$.*

(ii) *If $x > 0$, then $I^{h,e}(x, e) = e$.*

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In addition, these operators have proved their potential in edge detection through the fuzzy morphological gradient.

# Goal

To advance in the characterization of $(h, e)$-implications.

# From $(h, e)$-implications to some generalizations of Yager's implications

# Threshold horizontal generated implications

In 2013, Massanet and Torrens presented the structure of this family of fuzzy implication functions by using the threshold horizontal construction method.

## Theorem (**Massanet, Torrens (2012)**)

*Let $I_1$, $I_2$ be two fuzzy implication functions and $e \in (0,1)$. Then the binary function $I_{I_1-I_2} : [0,1]^2 \to [0,1]$, called the e-threshold horizontal generated implication from $I_1$ and $I_2$, defined as*

$$I_{I_1-I_2}(x,y) = \begin{cases} 1 & \text{if } x = 0, \\ e \cdot I_1\left(x, \dfrac{y}{e}\right) & \text{if } x > 0 \text{ and } y \leq e, \\ e + (1-e) \cdot I_2\left(x, \dfrac{y-e}{1-e}\right) & \text{if } x > 0 \text{ and } y > e, \end{cases}$$

*is a fuzzy implication function.*

# Threshold horizontal generated implications

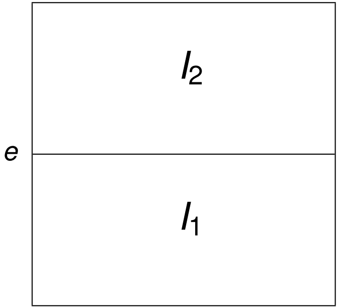


Figure: Structure of an $e$-threshold horizontal generated implication from $I_1$ and $I_2$.

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$$I_{f,e}(x, y) = f^{(-1)}\left(\frac{x}{e} \cdot f(y)\right), \quad x, y \in [0, 1]$$

with the understanding $0 \cdot \infty = 0$, is called an $(f, e)$*-generated operation* and $f$ its $f$-generator.

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## Definition

Let $g : [0,1] \to [0,+\infty]$ be a strictly increasing and continuous function with $g(0) = 0$ and $e \in (0,1)$. The function $I_{g,e} : [0,1]^2 \to [0,1]$ defined by

$$I_{g,e}(x,y) = g^{(-1)}\left(\frac{e}{x} \cdot g(y)\right), \quad x, y \in [0,1]$$

with the understanding $\frac{1}{0} = +\infty$ and $+\infty \cdot 0 = \infty$, is called a $(g,e)$*-generated operation* and $g$ its $g$-generator.

# Structure of $(h, e)$-implications

## Theorem (**Massanet, Torrens (2013)**)

*Let $I : [0,1]^2 \to [0,1]$ be a binary function and $e \in (0,1)$. Then the following statements are equivalent:*

(i) *$I$ is an $(h,e)$-implication with respect to $e$, that is, $I = I_{h,e}$.*

(ii) *There exist an $f$-generator with $f(0) = +\infty$ and a $g$-generator with $g(1) = +\infty$ such that $I$ is given by $I = I_{I_{f,e} - I_{g,e}}$.*

*Moreover, in this case generators $h$, $f$ and $g$ are related in the following way:*

$$f(x) = -h(ex), \quad g(x) = h(e + (1-e)x), \quad h(x) = \begin{cases} -f\left(\dfrac{x}{e}\right) & \text{if } x \le e, \\ g\left(\dfrac{x-e}{1-e}\right) & \text{if } x > e. \end{cases}$$

# Structure of $(h, e)$-implications

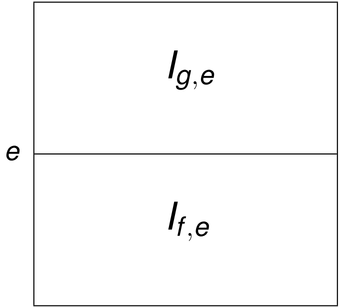
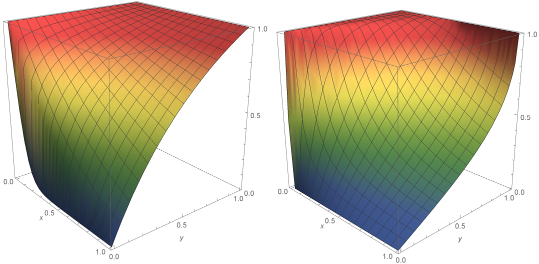



Figure: Structure of an $(h, e)$-implication.

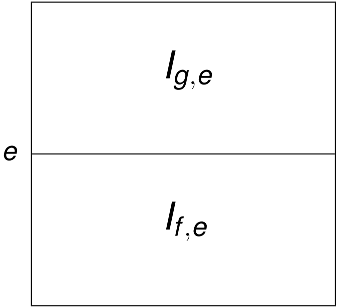


(a) $I_{f,\frac{1}{2}}$ with $f(x) = -\ln\left(\frac{x}{2-x}\right)$

(b) $I_{g,\frac{1}{2}}$ with $g(x) = \ln\left(\frac{1+x}{1-x}\right)$

Figure: Generating fuzzy implication functions $I_{f,e}$ and $I_{g,e}$.

# Structure of $(h, e)$-implications

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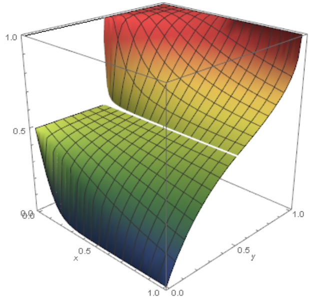


(a) $I^{h,\frac{1}{2}}$ with $h(x) = \ln\left(\frac{x}{1-x}\right)$

Figure: Generated $(h, e)$-implication $I^{h,e}$.

# Structure of $(h, e)$-implications

Therefore, it is straightforward to deduce that any possible characterization of $(h, e)$-implications must rely on the characterizations of $(f, e)$ and $(g, e)$-implications.

Indeed, in this paper, we have started by characterizing the $(f, e)$**-generated implications**.

# Characterization of $(f, e)$-implications

# Additional properties

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- If $f(0) < +\infty$, then the natural negation $N_{I_{f,e}}$ is given by

$$N_{I_{f,e}}(x) = \begin{cases} f^{-1}\left(\frac{x}{e} f(0)\right) & \text{if } x \leq e, \\ 0 & \text{if } x > e. \end{cases}$$

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- $I_{f,e}$ satisfies **(EP)** if and only if $f(0) = +\infty$.

# Law of importation

Special attention deserves the law of importation.

### Proposition

*Let $f$ be an $f$-generator and $e \in (0,1)$. Then $I_{f,e}$ does not satisfy* **(LI)** *with any t-norm.*

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However, they satisfy some modified versions of this property.

# Law of importation

## Definition

A fuzzy implication function $I$ is said to satisfy

1. the $(x, ey)$-*law of importation* with a t-norm $T$ for some $e \in (0,1)$, if

$$I(T(x,y),z) = I(x, I(ey,z)), \text{ for all } x,y,z \in [0,1]. \qquad (\mathbf{LI})_{\mathbf{x,ey}}$$

2. the $(ex, y)$-*law of importation* with a t-norm $T$ for some $e \in (0,1)$, if

$$I(T(x,y),z) = I(ex, I(y,z)), \text{ for all } x,y,z \in [0,1]. \qquad (\mathbf{LI})_{\mathbf{ex,y}}$$

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## Proposition

*Let $f$ be an $f$-generator and $e \in (0, 1)$. Then the following properties hold:*

1. *$I_{f,e}$ satisfies $(\mathbf{LI})_{x,ey}$ with respect to $T_{\mathbf{P}}$.*
2. *$I_{f,e}$ satisfies $(\mathbf{LI})_{ex,y}$ with respect to $T_{\mathbf{P}}$ if and only if $f(0) = +\infty$.*

# Characterization when $f(0) < +\infty$

## Theorem

*Let $I : [0,1]^2 \to [0,1]$ be a binary function and $e \in (0,1)$. Then the following statements are equivalent:*

(i) *$I$ is an $(f,e)$-generated implication with $f(0) < +\infty$.*

(ii) *$I$ satisfies $\textbf{(LI)}_{x,ey}$ with $T_{\mathbf{P}}$ and $N_I$ is a continuous fuzzy negation which is strictly decreasing in $(0,e)$ for some $e \in (0,1)$ and such that $N_I(e) = 0$.*

*Moreover, in this case the $f$-generator is given by*

$$f(x) = N_I^{(-1)}(x) = \begin{cases} N_I^{-1}(x) & \text{if } x > 0, \\ e & \text{if } x = 0. \end{cases}$$

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*Let* $I : [0,1]^2 \to [0,1]$ *be a binary function and* $e \in (0,1)$*. Then the following statements are equivalent:*

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(i) *$I$ is an $(f,e)$-generated implication with $f(0) = +\infty$.*

(ii) *$I$ satisfies $\textbf{(LI)}_{x,ey}$ and $\textbf{(LI)}_{ex,y}$ with respect to $T_{\mathbf{P}}$, $I$ is continuous except at $(0,0)$ and $I(x,y) = 1 \Leftrightarrow x = 0$ or $y = 1$.*

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## Theorem

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- (i) *$I$ is an $(f,e)$-generated implication with $f(0) = +\infty$.*
- (iii) *$I$ satisfies $\mathbf{(LI)}_{x,ey}$ and $\mathbf{(LI)}_{ex,y}$ with respect to $T_{\mathbf{P}}$, $N_I = N_{D_1}$ and there exists $k \in (0,1)$ such that*
  - *$h_k$ is continuous and strictly decreasing with $h_k(0) = 1$ and $h_k(e) = k$,*
  - *$h_\bullet^{-1}(k) : (0,k) \to [0,e]$ that assigns $h_y^{-1}(k)$ to some $y \in (0,k)$ is a well-defined, continuous and strictly increasing function satisfying $\lim_{y \to 0^+} h_y^{-1}(k) = 0$.*

*Moreover, in this case the $f$-generator is given by*

$$f(x) = \begin{cases} \dfrac{h_k^{-1}(x)}{e} & \text{if } k \le x \le 1, \\ \dfrac{e}{h_x^{-1}(k)} & \text{if } 0 < x < k, \\ +\infty & \text{if } x = 0. \end{cases}$$

# Independence between properties

<table>
<tr><th>Function $I$</th><th>(LI)$_{x,ey}$ with $T_{\mathbf{P}}$ for some $e \in (0,1)$</th><th>(LI)$_{ex,y}$ with $T_{\mathbf{P}}$ for some $e \in (0,1)$</th><th>$I(x,y) = 1 \Leftrightarrow x = 0$ or $y = 1$</th><th>$I$ cont $\setminus\{(0,0)\}$</th></tr>
<tr><td>$\begin{cases} \frac{xy}{e} & \text{if } xy \le e, \\ 1 & \text{if } xy > e. \end{cases}$</td><td bgcolor="green"></td><td bgcolor="red"></td><td bgcolor="red"></td><td bgcolor="red"></td></tr>
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# Conclusions and future work

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- two interesting modifications of the law of importation have been introduced.

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- To axiomatically characterize the $(h, e)$-implications.

Moreover, we want to study further the two modifications of the law of importation introduced here.

# Thank you for your attention!