On a New Construction Method of Fuzzy Sheffer Stroke Operation

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Outline

Motivation

Refrigerator Alarm

Let us consider a refrigerator with an alarm and two sensors.

The sensors measure:

- The opening degree of the door.
- The internal temperature.

Refrigerator Alarm

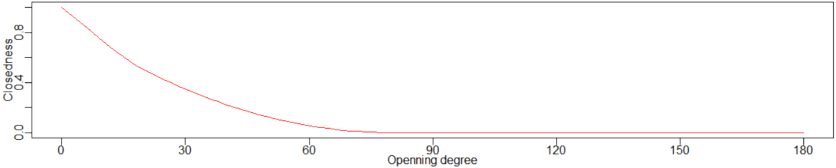
Let us consider a refrigerator with an alarm with different intensities and two sensors.

The sensors measure:

- The opening degree of the door.
- The internal temperature.

Refrigerator Alarm

- Let $A\colon X \to [0, 1]$ be the fuzzy set where X is the set of possible opening angles of the door. **(Closedness)**



Refrigerator Alarm

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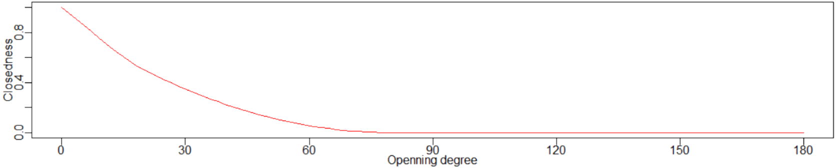
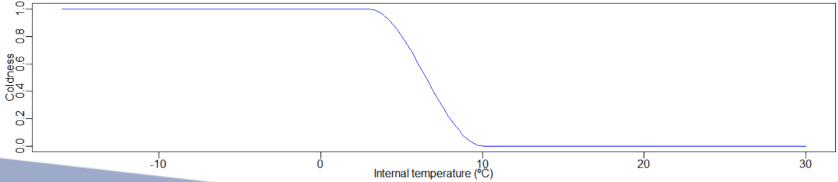


- Let $B\colon Y \to [0,1]$ be the fuzzy set where Y is the set of the possible internal temperatures. **(Coldness)**



Refrigerator Alarm

- Let $A\colon X \to [0,1]$ be the fuzzy set that represents **Closedness**.
- Let $B\colon Y \to [0,1]$ be the fuzzy set that represents **Coldness**.

We can model the alarm with the operator

$$F\colon [0,1] \times [0,1] \to [0,1]$$

- If $F(a,b) = 0$ the alarm does not sound.
- Whenever $F(a,b)$ increases, so does the intensity of the alarm.

Refrigerator Alarm

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- If the internal temperature of the fridge is hot, $B(y) = 0$, then the alarm should sound, so $F(a, 0) = 1$ for all $a \in [0, 1]$.

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- If the door is open, $A(x) = 0$, then the alarm should sound, so $F(0, b) = 1$ for all $b \in [0, 1]$.
- If the internal temperature of the fridge is hot, $B(y) = 0$, then the alarm should sound, so $F(a, 0) = 1$ for all $a \in [0, 1]$.
- When the door opening angle decreases, so does the intensity of the alarm. That is,

$$\text{if } a_1 \leq a_2, \text{ then } F(a_1, b) \geq F(a_2, b) \text{ for all } b \in [0, 1].$$

Refrigerator Alarm

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- If the door is open, $A(x) = 0$, then the alarm should sound, so $F(0, b) = 1$ for all $b \in [0, 1]$.
- If the internal temperature of the fridge is hot, $B(y) = 0$, then the alarm should sound, so $F(a, 0) = 1$ for all $a \in [0, 1]$.
- When the door opening angle decreases, so does the intensity of the alarm. That is,

  $$\text{if } a_1 \leq a_2, \text{ then } F(a_1, b) \geq F(a_2, b) \text{ for all } b \in [0, 1].$$

- On the other hand, when the internal temperature decreases, the intensity of the alarm also decreases. That is,

  $$\text{if } b_1 \leq b_2, \text{ then } F(a, b_1) \geq F(a, b_2) \text{ for all } a \in [0, 1].$$

Sheffer stroke

Table: Truth table for the classical Sheffer stroke.

p	q	$p \uparrow q$
0	0	1
0	1	1
1	0	1
1	1	0

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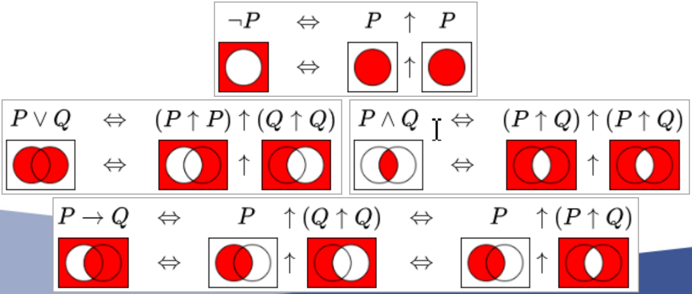

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1	1	0

Definition

A function $SH\colon [0,1]^2 \to [0,1]$ is called a **fuzzy Sheffer stroke operation** (or fuzzy Sheffer stroke) if it satisfies, for all $x, y, z \in [0,1]$, the following conditions:

(SH1) $SH(x,z) \geq SH(y,z)$ for $x \leq y$, i.e., $SH(\cdot, z)$ is non-increasing,

(SH2) $SH(x,y) \geq SH(x,z)$ for $y \leq z$, i.e., $SH(x, \cdot)$ is non-increasing,

(SH3) $SH(0,1) = SH(1,0) = 1$ and $SH(1,1) = 0$.

Examples of fuzzy Sheffer stroke

Example (The maximum fuzzy Sheffer stroke)

$$SH_{\max}(x,y) = \begin{cases} 0 & \text{if } (x,y) = (1,1) \\ 1 & \text{otherwise} \end{cases}$$

Examples of fuzzy Sheffer stroke

Example (The maximum fuzzy Sheffer stroke)

$$SH_{\max}(x,y) = \begin{cases} 0 & \text{if } (x,y) = (1,1) \\ 1 & \text{otherwise} \end{cases}$$

Example (The minimum fuzzy Sheffer stroke)

$$SH_{\min}(x,y) = \begin{cases} 0 & \text{if } (x,y) \in (0,1]^2 \\ 1 & \text{otherwise} \end{cases}$$

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$$SH_{\min}(x,y) = \begin{cases} 0 & \text{if } (x,y) \in (0,1]^2 \\ 1 & \text{otherwise} \end{cases}$$

Example

$$SH_M(x,y) = \max\{1-x, 1-y\}$$

Relation of other Fuzzy Connectives from Fuzzy Sheffer Stroke

Fuzzy Negation

Definition

A non-increasing function $N\colon [0,1] \to [0,1]$ is called a **fuzzy negation** if

$$N(0) = 1 \qquad \text{and} \qquad N(1) = 0.$$

Fuzzy Negation

Definition

A non-increasing function $N\colon [0,1] \to [0,1]$ is called a **fuzzy negation** if

$$N(0) = 1 \qquad \text{and} \qquad N(1) = 0.$$

Moreover, a fuzzy negation N is called

1. **strict** if it is strictly decreasing and continuous;
2. **strong** if it is an involution, i.e., $N(N(x)) = x$ for all $x \in [0,1]$.

Natural negations of fuzzy Sheffer stroke

Definition

Let SH be a Sheffer stroke operation, we define the following three negations:

1. The ***left natural negation*** of SH defined by

$$N^{l}_{SH}(x) = SH(x, 1) \text{ for all } x \in [0, 1].$$

2. The ***right natural negation*** of SH defined by

$$N^{r}_{SH}(x) = SH(1, x) \text{ for all } x \in [0, 1].$$

3. The ***diagonal natural negation*** of SH defined by

$$N^{d}_{SH}(x) = SH(x, x) \text{ for all } x \in [0, 1].$$

Natural negations of some Sheffer strokes

	$\mathbf{N^l_{SH}}$	$\mathbf{N^r_{SH}}$	$\mathbf{N^d_{SH}}$
$\mathbf{SH_{Min}}$	N_{D_2}	N_{D_2}	N_{D_2}
$\mathbf{SH_{Max}}$	N_{D_1}	N_{D_1}	N_{D_1}
$\mathbf{SH_M}$	N_C	N_C	N_C

$$N_{D_1}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases} \qquad N_{D_2}(x) = \begin{cases} 1 & \text{if } x < 1 \\ 0 & \text{if } x = 1 \end{cases}$$

$$N_C(x) = 1 - x \qquad \text{for all } x \in [0,1]$$

Fuzzy Conjunction

Definition

A function $C\colon [0,1]^2 \to [0,1]$ is called a **fuzzy conjunction** if it satisfies, for all $x, y, z \in [0,1]$, the following conditions:

(C1) $C(x,y) \le C(z,y)$ for $x \le z$, i.e., $C(\cdot, y)$ is non-decreasing,

(C2) $C(x,y) \le C(x,z)$ for $y \le z$, i.e., $C(x, \cdot)$ is non-decreasing,

(C3) $C(0,1) = C(1,0) = 0$ and $C(1,1) = 1$.

Construction of conjunctions

To generate conjunctions, we apply the tautologies from classical logic:

$$p \wedge q \equiv ((p \uparrow q) \uparrow (p \uparrow q)),$$

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Theorem

Let SH be a fuzzy Sheffer stroke. Then, the following function

$$C(x, y) = SH(SH(x, y), SH(x, y)), \qquad x, y \in [0, 1]$$

is a fuzzy conjunction.

Fuzzy conjunctions from some Sheffer strokes

Example (Conjunction from SH_{Max})

$$SH_{Max}(SH_{Max}(x,y), SH_{Max}(x,y)) = \begin{cases} 1 & \text{if } (x,y) = (1,1) \\ 0 & \text{otherwise} \end{cases}$$

Fuzzy conjunctions from some Sheffer strokes

Example (Conjunction from SH_{Max})

$$SH_{Max}(SH_{Max}(x,y), SH_{Max}(x,y)) = \begin{cases} 1 & \text{if } (x,y) = (1,1) \\ 0 & \text{otherwise} \end{cases} = C_{\min}(x,y)$$

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Example (Conjunction from SH_{Min})

$$SH_{Min}(SH_{Min}(x,y), SH_{Min}(x,y)) = \begin{cases} 1 & \text{if } (x,y) \in (0,1]^2 \\ 0 & \text{otherwise} \end{cases}$$

Fuzzy conjunctions from some Sheffer strokes

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Example (Conjunction from SH_M)

$$SH_M(SH_M(x,y), SH_M(x,y)) = \min\{x,y\}$$

Fuzzy conjunctions from some Sheffer strokes

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Example (Conjunction from SH_{M})

$$SH_M(SH_M(x,y), SH_M(x,y)) = \min\{x,y\} = T_{\mathbf{M}}(x,y)$$

Fuzzy Disjunction

Definition

A function $D\colon [0,1]^2 \to [0,1]$ is called a **fuzzy disjunction** if it satisfies, for all $x, y, z \in [0,1]$, the following conditions:

(D1) $D(x,y) \leq D(z,y)$ for $x \leq z$, i.e., $D(\cdot, y)$ is non-decreasing,

(D2) $D(x,y) \leq D(x,z)$ for $y \leq z$, i.e., $D(x, \cdot)$ is non-decreasing,

(D3) $D(0,1) = D(1,0) = 1$ and $D(0,0) = 0$.

Construction of disjunctions

To generate disjunctions, we apply the tautologies from classical logic:

$$p \vee q \equiv ((p \uparrow p) \uparrow (q \uparrow q)).$$

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Theorem

Let SH be a fuzzy Sheffer stroke. Then, the following function

$$D(x, y) = SH(SH(x, x), SH(y, y)), \qquad x, y \in [0, 1]$$

is a fuzzy disjunction.

Fuzzy disjunctions from some Sheffer strokes

Example (Disjunction from SH_{Max})

$$SH_{Max}(SH_{Max}(x,x), SH_{Max}(y,y)) = \begin{cases} 0 & \text{if } (x,y) \in [0,1)^2 \\ 1 & \text{otherwise} \end{cases}$$

Fuzzy disjunctions from some Sheffer strokes

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Example (Disjunction from SH_M)

$$SH_M(SH_M(x,x), SH_M(y,y)) = \max\{x,y\}$$

Fuzzy disjunctions from some Sheffer strokes

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Example (Disjunction from SH_M)

$$SH_M(SH_M(x,x), SH_M(y,y)) = \max\{x,y\} = S_{\mathbf{M}}(x,y)$$

t-norms and t-conorms

The most studied conjunctions

- An associative, commutative and increasing operation $T\colon [0,1]^2 \to [0,1]$ is called a **t-norm** if it has the neutral element 1.

The most studied disjunctions

- An associative, commutative and increasing operation $S\colon [0,1]^2 \to [0,1]$ is called a **t-conorm** if it has the neutral element 0.

Additional properties

Definition

A function $SH\colon [0,1]^2 \to [0,1]$ is called a **fuzzy Sheffer stroke operation** (or fuzzy Sheffer stroke) if it satisfies, for all $x, y, z \in [0,1]$, the following conditions:

(SH1) $SH(x, z) \geq SH(y, z)$ for $x \leq y$, i.e., $SH(\cdot, z)$ is non-increasing,

(SH2) $SH(x, y) \geq SH(x, z)$ for $y \leq z$, i.e., $SH(x, \cdot)$ is non-increasing,

(SH3) $SH(0, 1) = SH(1, 0) = 1$ and $SH(1, 1) = 0$.

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(SH3) $SH(0,1) = SH(1,0) = 1$ and $SH(1,1) = 0$.

(SH4) $SH(SH(x,x), SH(x,x)) = x$, for all $x \in [0,1]$,

(SH5) $SH(1,x) = SH(x,x)$, for all $x \in [0,1]$,

(SH6) $SH(x,y) = SH(y,x)$, for all $x, y \in [0,1]$,

(SH7) $SH(x, SH(SH(y,z), SH(y,z))) =$
$SH(SH(SH(x,y), SH(x,y)), z)$, for all $x, y, z \in [0,1]$.

Construction of t-norms and t-conorms

(SH4) $SH(SH(x,x),SH(x,x)) = x$, for all $x \in [0,1]$,

(SH5) $SH(1,x) = SH(x,x)$, for all $x \in [0,1]$,

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(SH7) $SH(x, SH(SH(y,z), SH(y,z))) =$
$SH(SH(SH(x,y), SH(x,y)), z)$, for all $x, y, z \in [0,1]$.

Theorem

*Let SH be a fuzzy Sheffer stroke that satisfies (**SH4**). Then, the following function*

$$T(x,y) = SH(SH(x,y), SH(x,y)), \qquad x, y \in [0,1]$$

*is a t-norm if and only if SH satisfies additionally (**SH5**), (**SH6**) and (**SH7**).*

Construction of t-norms and t-conorms

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Theorem

*Let SH be a fuzzy Sheffer stroke that satisfies (**SH4**). Then, the following function*

$$S(x,y) = SH(SH(x,x), SH(y,y)), \qquad x, y \in [0,1]$$

*is a t-conorm if and only if SH satisfies additionally (**SH5**), (**SH6**) and (**SH7**).*

Fuzzy Implication

Definition

A function $I\colon [0,1]^2 \to [0,1]$ is called a **fuzzy implication function** if it satisfies, for all $x, y, z \in [0,1]$, the following conditions:

(I1) $I(x,z) \geq I(y,z)$ for $x \leq y$, i.e., $I(\cdot, z)$ is non-increasing,

(I2) $I(x,y) \leq I(x,z)$ for $y \leq z$, i.e., $I(x, \cdot)$ is non-decreasing,

(I3) $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.

Construction of implications

In classical logic, the two-valued implication can be presented using only Sheffer stroke operation in two ways:

$$p \to q \equiv p \uparrow (q \uparrow q) \equiv \neg(p \wedge \neg(q \wedge q)),$$

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Theorem

Let SH be a fuzzy Sheffer stroke. Then the function I defined by

$$I(x, y) = SH(x, SH(y, y)), \qquad x, y \in [0, 1],$$

is a fuzzy implication function.

Fuzzy implications from some Sheffer strokes

Example (Implication from SH_{Max})

$$SH_{Max}(x, SH_{Max}(y, y)) = \begin{cases} 0 & \text{if } x = 1,\ y < 1 \\ 1 & \text{otherwise} \end{cases}$$

Fuzzy implications from some Sheffer strokes

Example (Implication from SH_{Max})

$$SH_{Max}(x, SH_{Max}(y, y)) = \begin{cases} 0 & \text{if } x = 1,\ y < 1 \\ 1 & \text{otherwise} \end{cases}$$

Example (Implication from SH_{Min})

$$SH_{Min}(x, SH_{Min}(y, y)) = \begin{cases} 0 & \text{if } x > 0,\ y = 0 \\ 1 & \text{otherwise} \end{cases}$$

Fuzzy implications from some Sheffer strokes

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Example (Implication from SH_M)

$$SH_M(x, SH_M(y, y)) = \max\{1 - x, y\}$$

Fuzzy implications from some Sheffer strokes

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Example (Implication from SH_{Min})

$$SH_{Min}(x, SH_{Min}(y, y)) = \begin{cases} 0 & \text{if } x > 0,\ y = 0 \\ 1 & \text{otherwise} \end{cases}$$

Example (Implication from SH_{M})

$$SH_M(x, SH_M(y, y)) = \max\{1 - x, y\} = I_{KD}(x, y)$$

Construction of implications

In classical logic, the two-valued implication can be presented using only Sheffer stroke operation in two ways:

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Theorem

Let SH be a fuzzy Sheffer stroke. Then the function $I^\star$ defined by

$$I^\star(x, y) = SH(x, SH(x, y)), \qquad x, y \in [0, 1],$$

*satisfies (**I2**) and (**I3**).*

Construction of implications

$$SH_M(x, y) = \max\{1 - x, 1 - y\}$$

Example

$$I_M^{\star}(x, y) = SH_M(x, SH_M(x, y)) = \begin{cases} \max\{x, 1 - x\} & \text{if } x \leq y \\ \max\{1 - x, y\} & \text{if } x > y \end{cases}$$

Construction of implications

$$SH_M(x,y) = \max\{1-x, 1-y\}$$

Example

$$I^{\star}_M(x,y) = SH_M(x, SH_M(x,y)) = \begin{cases} \max\{x, 1-x\} & \text{if } x \leq y \\ \max\{1-x, y\} & \text{if } x > y \end{cases}$$

$$I^{\star}_M(0.8, 1) = \max\{0.8, 1-0.8\} = 0.8$$

$$I^{\star}_M(0.9, 1) = \max\{0.9, 1-0.9\} = 0.9$$

$$I^{\star}_M(0.8, 1) < I^{\star}_M(0.9, 1)$$

Construction of implications

$$SH_M(x,y) = \max\{1-x, 1-y\}$$

Example

$$I_M^\star(x,y) = SH_M(x, SH_M(x,y)) = \begin{cases} \max\{x, 1-x\} & \text{if } x \le y \\ \max\{1-x, y\} & \text{if } x > y \end{cases}$$

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$I_M^\star$ does not satisfy (**I1**)!

Characterization of fuzzy Sheffer stroke

In classical logic, Sheffer stroke is the negation of the conjunction (NAND),

$$p \uparrow q = \neg(p \wedge q).$$

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Theorem (Theorem of characterization of fuzzy Sheffer stroke)

Let $SH \colon [0,1]^2 \to [0,1]$ be a binary operation. Then the following statements are equivalent:

1. *SH is a fuzzy Sheffer stroke.*
2. *There exist a fuzzy conjunction C and a strict fuzzy negation N such that $SH(x,y) = N(C(x,y))$ for all $x, y \in [0,1]$.*

Moreover, in this case, $C(x,y) = N^{-1}(SH(x,y))$ for all $x, y \in [0,1]$.

Characterization of fuzzy Sheffer stroke

$$SH(x, y) = N(C(x, y)) \text{ for all } x, y \in [0, 1].$$

Remark

1. The representation of a fuzzy Sheffer stroke is not unique. Indeed, any strict fuzzy negation N can be chosen. However, fixed a strict fuzzy negation N, the fuzzy conjunction C is unique.
2. Whenever one of the natural negations of the fuzzy Sheffer stroke is strict, it can be considered to represent the fuzzy Sheffer stroke. In this case, both the fuzzy negation and the fuzzy conjunction are defined from the expression of SH.

Fuzzy Sheffer strokes constructions

Constructions of fuzzy Sheffer strokes

We construct some Sheffer strokes from the classic fuzzy negation and some different t-norms.

- The minimum t-norm $T_{\mathbf{M}}(x, y) = \min\{x, y\}$.
- The product t-norm $T_{\mathbf{P}}(x, y) = xy$.
- The Łukasiewicz t-norm $T_{\mathbf{LK}}(x, y)$.

Constructions of fuzzy Sheffer strokes

$$\text{Sheffer stroke from } \begin{cases} \text{the minimum t-norm } T_{\mathbf{M}}(x, y) = \min\{x, y\} \text{ and} \\ \\ \text{the classical negation } N_C(x) = 1 - x \end{cases}$$

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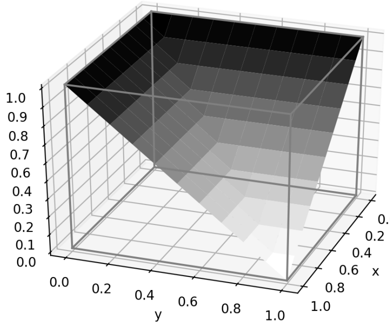
$$SH_M(x,y) = 1 - T_{\mathbf{M}}(x,y)$$

$$SH_M(x,y) = \max\{1-x, 1-y\}$$



Constructions of fuzzy Sheffer strokes

$$\text{Sheffer stroke from} \begin{cases} \text{the product t-norm } T_{\mathbf{P}}(x,y) = xy \text{ and} \\ \\ \text{the classical negation } N_C(x) = 1 - x \end{cases}$$

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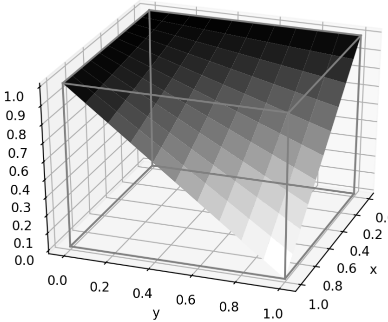
$$SH_P(x,y) = 1 - T_{\mathbf{P}}(x,y)$$

$$SH_P(x,y) = 1 - xy$$



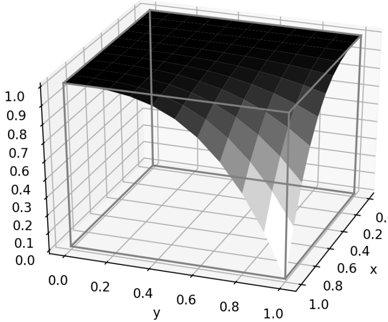
Constructions of fuzzy Sheffer strokes

$$\text{Sheffer stroke from} \begin{cases} \text{the general fuzzy conjunction } C_P^k(x,y) = (xy)^k \text{ and} \\ \\ \text{the classical negation } N_C(x) = 1 - x \end{cases}$$

Constructions of fuzzy Sheffer strokes

Sheffer stroke from $\begin{cases} \text{the general fuzzy conjunction } C_P^k(x,y) = (xy)^k \text{ and} \\ \text{the classical negation } N_C(x) = 1 - x \end{cases}$

$$SH_P^k(x,y) = 1 - C_P^k(x,y)$$
$$SH_P^k(x,y) = 1 - (xy)^k$$



Constructions of fuzzy Sheffer strokes

$$\text{Sheffer stroke from} \begin{cases} \text{the Łukasiewicz t-norm } T_{\mathbf{LK}}(x, y) \text{ and} \\ \\ \text{the classical negation } N_C(x) = 1 - x \end{cases}$$

Constructions of fuzzy Sheffer strokes

$$\text{Sheffer stroke from} \begin{cases} \text{the Łukasiewicz t-norm } T_{\mathbf{LK}}(x,y) \text{ and} \\ \\ \text{the classical negation } N_C(x) = 1 - x \end{cases}$$

$$T_{\mathbf{LK}}(x,y) = \max\{x+y-1, 0\}$$

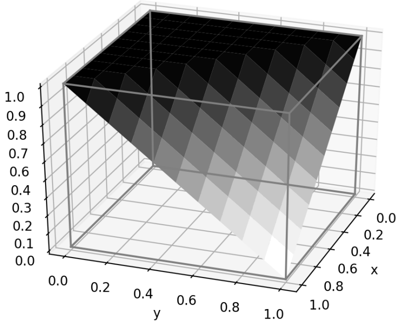
Constructions of fuzzy Sheffer strokes

Sheffer stroke from $\begin{cases} \text{the Łukasiewicz t-norm } T_{\mathbf{LK}}(x,y) \text{ and} \\ \text{the classical negation } N_C(x) = 1 - x \end{cases}$

$$T_{\mathbf{LK}}(x,y) = \max\{x+y-1, 0\}$$

$$SH_{LK}(x,y) = 1 - T_{\mathbf{LK}}(x,y)$$

$$SH_{LK}(x,y) = \min\{2-x-y, 1\}$$



Natural negations of fuzzy Sheffer strokes given

	$\mathbf{N}^{\mathbf{l}}_{\mathbf{SH}}$	$\mathbf{N}^{\mathbf{r}}_{\mathbf{SH}}$	$\mathbf{N}^{\mathbf{d}}_{\mathbf{SH}}$
SH_M	N_C	N_C	N_C
SH_P	N_C	N_C	$1 - x^2$
SH_P^k	$1 - x^k$	$1 - x^k$	$1 - x^{2k}$
SH_{LK}	N_C	N_C	$\begin{cases} 1, & \text{if } x \leq 0.5, \\ 2 - 2x, & \text{otherwise.} \end{cases}$
SH_D	N_C	N_C	greatest fuzzy negation N_{D_2}

Construction of Fuzzy Sheffer Stroke from univalued functions

Additive generators

Definition

Let $f\colon [0,1] \to [0,+\infty]$
be a **decreasing function** with

$$f(0) = +\infty \quad \text{and} \quad f(1) = 0,$$

and let $g\colon [0,+\infty] \to [0,1]$
be an **increasing function** with

$$g(0) = 0 \quad \text{and} \quad g(+\infty) = 1.$$

The operator $SH_{f,g}\colon [0,1]^2 \to [0,1]$ defined by

$$SH_{f,g}(x,y) = g(f(x) + f(y))$$

is called an (f,g)-Sheffer stroke.

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In this case, the pair (f,g) is called the pair of additive generators of $SH_{f,g}$.

Theorem

$SH_{f,g}$ is always a fuzzy Sheffer Stroke.

Sheffer Stroke from non-continuous functions

Example

Let us consider the non-continuous functions

$$f(x) = \begin{cases} +\infty, & \text{if } x < 1, \\ 0, & \text{if } x = 1, \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{otherwise,} \end{cases}$$

which satisfy the requirements of previous definition.

Sheffer Stroke from non-continuous functions

Example

Let us consider the non-continuous functions

$$f(x) = \begin{cases} +\infty, & \text{if } x < 1, \\ 0, & \text{if } x = 1, \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{otherwise}, \end{cases}$$

which satisfy the requirements of previous definition.Then we obtain the fuzzy Sheffer stroke given by

$$SH_{\max}(x, y) = SH_{f,g}(x, y) = \begin{cases} 0, & \text{if } (x, y) = (1, 1), \\ 1, & \text{otherwise}, \end{cases}$$

which is the maximum fuzzy Sheffer stroke.

Sheffer Stroke from non-continuous functions

Example

Let us consider the non-continuous functions

$$f(x) = \begin{cases} +\infty & \text{if } x = 0 \\ 1 - x & \text{if } x > 0 \end{cases} \qquad \text{and} \qquad g(x) = \min\{x, 1\}$$

which satisfy the requirements of previous definition.

Sheffer Stroke from non-continuous functions

Example

Let us consider the non-continuous functions

$$f(x) = \begin{cases} +\infty & \text{if } x = 0 \\ 1 - x & \text{if } x > 0 \end{cases} \qquad \text{and} \qquad g(x) = \min\{x, 1\}$$

which satisfy the requirements of previous definition. Then we obtain the fuzzy Sheffer stroke given by

$$SH_{LK}(x, y) = SH_{f,g}(x, y) = \min\{2 - x - y, 1\}$$

which is the fuzzy Sheffer stroke obtained from the classic fuzzy negation and the Łukasiewicz t-norm.

T-norms from Sheffer strokes

Proposition

Let (f, g) be a pair of additive generators of an (f, g)-Sheffer stroke with g a continuous and strictly increasing function and let N be a strict fuzzy negation. Then the following statements are equivalent:

1. $N^{-1} \circ SH_{f,g}$ *is a t-norm.*
2. $f = g^{-1} \circ N$.

In this case, the expression of the fuzzy Sheffer stroke is given by

$$SH_{g,N}(x, y) = g\left(g^{-1}(N(x)) + g^{-1}(N(y))\right)$$

for all $x, y \in [0, 1]$.

Properties of the Sheffer strokes $SH_{g,N}$

$$SH_{g,N}(x,y) = g\left(g^{-1}(N(x)) + g^{-1}(N(y))\right)$$

(SH4) $SH(SH(x,x), SH(x,x)) = x$, for all $x \in [0,1]$.

Proposition

Let $g\colon [0,+\infty] \to [0,1]$ be a continuous and strictly increasing function and let N be a strict fuzzy negation N. Then the following statements are equivalent:

1. *$SH_{g,N}$ satisfies (**SH4**).*
2. *Exist an automorphism $\varphi\colon [0,1] \to [0,1]$ such that $N(x) = g\left(\frac{g^{-1}(N_C)_{\varphi}(x)}{2}\right)$ where N_C is a classical negation $N_C(x) = 1 - x$.*

Properties of the Sheffer strokes $SH_{g,N}$

$$SH_{g,N}(x,y) = g\left(g^{-1}(N(x)) + g^{-1}(N(y))\right)$$

(SH5) $SH(1,x) = SH(x,x)$, for all $x \in [0,1]$.

Proposition

*Let $g\colon [0,+\infty] \to [0,1]$ be a continuous and strictly increasing function and let N be a strict fuzzy negation N. Then $SH_{g,N}$ never satisfies (**SH5**).*

Properties of the Sheffer strokes $SH_{g,N}$

$$SH_{g,N}(x, y) = g\left(g^{-1}(N(x)) + g^{-1}(N(y))\right)$$

(SH6) $SH(x, y) = SH(y, x)$, for all $x, y \in [0, 1]$.

Proposition

*$SH_{g,N}$ always satisfies (**SH6**)*

Properties of the Sheffer strokes $SH_{g,N}$

$$SH_{g,N}(x,y) = g\left(g^{-1}(N(x)) + g^{-1}(N(y))\right)$$

(SH7) $SH(x, SH(SH(y,z), SH(y,z))) =$
$SH(SH(SH(x,y), SH(x,y)), z)$, for all $x, y, z \in [0,1]$.

Open Problem

It remains still an open problem whether there exist some g and N such that $SH_{g,N}$ satisfies (**SH7**).

Conclusions and future work

Conclusions

- We have generalized the Sheffer stroke operator to the fuzzy logic framework.
- We have studied the different methods of construction of other connectives from this one.
- We have given some different construction methods to generate fuzzy Sheffer strokes with different properties.

Future Work

- Study the additional properties of the implications

  $$I(x, y) = SH(x, SH(y, y))$$

  and their intersection with other known implication families.

- Define, characterize and study the operator **Pierce Arrow**, and also its relationship with other connectives.

Thank you for your attention!