Natural negations associated to discrete t-subnorms

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Abstract. In the last decades, operations defined on finite chains, usually called discrete operations, have experienced a peak of interest due to their applications in many fields. One of such operations are discrete t-subnorms which are a generalization of discrete t-norms, which have a paramount importance in applications dealing with linguistic labels. In this paper, the natural associated negations of discrete t-subnorms are deeply studied. From this study, several insights into the structure of these operators are presented and some properties are studied. Specifically, the discrete negations which can be the natural associated negation of a discrete t-subnorm are characterized in some particular cases. Throughout the paper the concepts of weak and symmetrical negation, which turn out to be equivalent in the discrete case, play a key role.

Keywords: discrete aggregation function, discrete t-subnorm, natural negation

1 Introduction

In many practical situations in which the range of computations and reasoning must be reduced to a finite set of possible values, often qualitative, the fuzzy linguistic approach is the adequate framework to model the information. This is because in this case, the qualitative terms used by experts are usually represented via linguistic variables instead of numerical values. Whenever this applies, linguistic variables are often interpreted to take values on totally ordered scales such as:

$$L = \{\text{Extremely Bad}, \text{Very Bad}, \text{Bad}, \text{Fair}, \text{Good}, \text{Very Good}, \text{Extremely Good}\},$$

which can be all represented by the finite chain $L_n = \{0, 1, \ldots, n\}$. Consequently, many researchers have focused their efforts to study operations defined on L_n,

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or discrete operations for short (see [7, 8, 11, 18] and specially [23] as the pioneer papers on this topic).

Among the whole set of discrete operations, discrete aggregation functions stand out due to the necessity to merge some data into a representative output. From decision making and subjective evaluations to image analysis and pattern recognition, the number of applications of aggregation functions is growing constantly as the number of different and complementary monographs that have been published in last years corroborate (see [2, 3, 12]). In most cases, the study on aggregation functions is performed assuming the smoothness property, which is usually considered as the counterpart of continuity for this kind of operations, or the equivalent 1-Lipschitz property. In this topic, many families of discrete aggregation functions have already been studied or even characterized. For instance, smooth t-norms and t-conorms were characterized in [24] (see also [23]), smooth t-subnorms in [20], weighted ordinal means in [16], uninorms in $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ and nullnorms in [18], discrete idempotent uninorms in [6], uninorms and nullnorms without the commutative property in [19] and [8], respectively, copulas in [22] and quasi-copulas in [1].

In addition to the proper division on families, discrete aggregation functions can be divided depending on their relationship with the minimum and the maximum leading to four classes: conjunctive when they lie under the minimum, disjunctive when they lie over the maximum, averaging or compensatory when they lie between the minimum and the maximum, and mixed otherwise. In this paper, we will focus on the class of conjunctive discrete aggregation functions and in particular, on the family of discrete t-subnorms. These operations generalize the well-known family of discrete t-norms (see [7, 15]) and can be viewed as a particular case of an order topological semigroup [5]. T-subnorms on $[0, 1]$ play a key role in the ordinal sum based construction of left-continuous t-norms and in other construction methods (see [14]). Other studies on these operations related to their representation through additive and multiplicative generators [9, 21, 25] and the fulfilment of some properties such as cancellativity [17] or lipschitzianity [10] have also supported the importance of t-subnorms.

Recently, one line of research on t-subnorms on $[0, 1]$ has been devoted to their associated natural negation. The concept of the associated natural negation was already studied in [4] for left-continuous t-norms, but it is in [13] where a complete study of t-subnorms with strong associated natural negation was performed. There, it was proved that such t-subnorms are in fact t-norms and the relationships between different algebraic and analytic properties such as Archimedeanness, conditional cancellativity, left-continuity and nilpotent elements were studied. Thus, within this line of research, in this paper we want to perform a similar study for discrete t-subnorms. The natural associated negations of these discrete operations will be deeply studied and several insights into the structure of these operators will be presented.

The paper is organized as follows. Section 2 is devoted to some preliminaries on discrete operations, especially discrete t-subnorms and discrete t-norms, in order to make the paper as self-contained as possible. Section 3 deals with the study of weak negations and symmetrical negations, two subfamilies of discrete negations which are proved to be equivalent in this context. In Section 4, the concept of 0-function of a discrete t-subnorm is introduced. From this concept, the cases when this 0-function becomes a discrete negation are fully determined. Moreover, among other important properties, it is proved that weak negations are the only discrete negations which are the natural associated negations of a discrete t-norm. Finally, the paper ends with Section 5 devoted to some conclusions and future work.

2 Preliminaries

We will suppose the reader to be familiar with the basic theory of aggregation functions, fuzzy negations, t-norms (see [15]) and also with discrete t-norms, that is, t-norms defined on a finite chain (see [24]). We only recall the definitions and facts that will be used in the paper.

It is clear (see [24]) that for the study of binary aggregation functions all finite chains with the same number of elements are equivalent. Thus, we will use the most simple one with $n+1$ elements:

$$L_n = \{0, 1, 2, \ldots, n\}$$

and, for all $a, b \in L_n$ with $a \leq b$, we will use also the notation $[a, b]$ to denote the finite subchain given by $[a, b] = \{x \in L_n \mid a \leq x \leq b\}$.

Definition 1 ([24]).

- *A function $f : L_n \to L_n$ is said to be* smooth *if it satisfies $\mid f(x) - f(x-1) \mid \leq 1$ for all $x \in L_n$ with $x \geq 1$.*
- *A binary operation F on L_n is said to be* smooth *when each one of its vertical and horizontal sections are smooth.*

The importance of the smoothness condition lies in the fact that it is generally used as a discrete counterpart of continuity on [0,1], because it is equivalent to the divisibility property (for a t-norm T, $x \leq y$ if and only if there is $z \in L_n$ such that $T(y, z) = x$), see again [24].

Proposition 1 ([24]). *The only smooth (equivalently strong or strictly decreasing) negation on L_n is the classical negation given by*

$$N(x) = n - x \quad \text{for all} \quad x \in L_n.$$

Proposition 2 ([24]). *Let $0 = a_0 < a_1 < \ldots < a_{m-1} < a_m = n$ be $m+1$ elements in L_n and let T_i be a t-norm on the chain $[a_{i-1}, a_i]$ for all $i = 1, \ldots, m$. Then the binary operation on L_n given by*

$$T(x, y) = \begin{cases} T_i(x, y) & \text{if there is an } i \text{ such that } a_{i-1} \leq x, y \leq a_i, \\ \min\{x, y\} & \text{otherwise,} \end{cases}$$

is always a t-norm on L_n usually called the ordinal sum of t-norms $T_1, \ldots, T_m$.

Proposition 3 ([24]). *There exists one and only one Archimedean smooth t-norm on L_n which is given by*

$$T(x,y) = \max\{0, x+y-n\} \tag{1}$$

and it is usually known as the Łukasiewicz t-norm.

Moreover, all smooth t-norms are characterized as ordinal sums of Łukasiewicz t-norms as follows.

Proposition 4 ([24]). *A t-norm T on L_n is smooth if and only if there exists a natural number m with $1 \leq m \leq n$ and a subset J of L_n,*

$$J = \{0 = a_0 < a_1 < \ldots < a_{m-1} < a_m = n\}$$

such that T is given by

$$T(x,y) = \begin{cases} \max\{a_k, x+y-a_{k+1}\} & \text{if there is } a_k \in J \text{ with } a_k \leq x, y \leq a_{k+1}, \\ \min\{x,y\} & \text{otherwise.} \end{cases}$$

A more general concept than t-norm is the one of t-subnorm.

Definition 2. *Let $T : L_n^2 \to L_n$ be a binary operation on L_n. Then T is said to be a* t-subnorm *when T is associative, commutative, non-decreasing in each variable and such that $T(x,y) \leq \min\{x,y\}$ for all $x, y \in L_n$.*

Obviously, any t-norm on L_n is also a t-subnorm but not vice versa. For instance, the weakest t-subnorm on L_n is the zero t-subnorm ($T(x,y) = 0$ for all $x, y \in L_n$) which clearly is not a t-norm.

3 Some properties on discrete negations

Since the only smooth negation on L_n is the classical one, one can search for alternative possibilities other than the classical negation $N(x) = n - x$. In the non-smooth case we can find many other possibilities for discrete negations as follows. The following definitions have been adapted from similar ones in [4] and [6].

Definition 3. *Let $N : L_n \to L_n$ be a discrete negation.*

- *N is said to be a* weak negation *when $x \leq N^2(x)$ for all $x \in L_n$.*
- *N is said to be* symmetrical *when the set*

$$F_N = \{(n,0)\} \cup \{(x,y) \in L_n^2 \mid N(x+1) \leq y \leq N(x)\}$$

is symmetrical, that is, $(x,y) \in F_N$ if and only if $(y,x) \in F_N$.

In the case of the interval $[0,1]$, weak and symmetrical negations do not coincide in general and they coincide only when the negation N is left-continuous (see [4]). The following example illustrates this fact.

Example 1. Roughly speaking, symmetrical negations are those negations N whose graph is symmetrical with respect to the identity function. Thus, to each possible constant region of N it corresponds a discontinuity point and vice versa (see again [4]). If we do not have left-continuity in these points the property $x \leq N^2(x)$ can fail. See for instance the negation given by

$$N(x) = \begin{cases} 1 & \text{if } x \leq 0.25, \\ 1.25 - x & \text{if } 0.25 < x < 1, \\ 0 & \text{if } x = 1. \end{cases}$$

It can be easily proved that this negation N is symmetrical, but clearly it is not a weak negation because for all x such that $0 < x \leq 0.25$ we have $N^2(x) = N(1) = 0 < x$.

In the discrete case both concepts always coincide as it is shown in the next proposition. The proof can be easily adapted to the discrete case from the one of Lemma 2 in [6]. However we include it here for the sake of completeness.

Proposition 5. *Let $N : L_n \to L_n$ be a discrete negation. The following items are equivalent:*

i) N is symmetrical.
ii) N is a weak negation.
iii) For all $(x, y) \in L_n^2$ it holds that:

$$y \leq N(x) \quad \Longleftrightarrow \quad x \leq N(y).$$

Proof. $(i) \Longrightarrow (ii)$. For all $x \in L_n$ we have by definition that $(x, N(x)) \in F_N$. Since N is symmetrical it must be also $(N(x), x) \in F_N$ and this implies that $x \leq N(N(x))$. That is, N is a weak negation.

$(ii) \Longrightarrow (iii)$. Consider $(x, y) \in L_n^2$ such that $x \leq N(y)$, the decreasingness of N implies that $N(x) \geq N(N(y)) \geq y$. Similarly, from $y \leq N(x)$ it follows that $x \leq N(y)$.

$(iii) \Longrightarrow (i)$. We want to prove that F_N is symmetrical. Consider $(x, y) \in F_N$, then $N(x+1) \leq y \leq N(x)$ and hence $x \leq N(y)$. On the other hand, if we suppose that $N(y+1) > x$, then

$$x + 1 \leq N(y + 1) \quad \Longrightarrow \quad y + 1 \leq N(x + 1) \quad \Longrightarrow \quad y < N(x + 1)\,,$$

which contradicts the fact that $(x, y) \in F_N$. We conclude that $N(y + 1) \leq x$ and thus $(y, x) \in F_N$, proving that F_N is symmetrical. □

There are many examples of discrete weak (or symmetrical) negations on L_n. In the following example we present a parametric family of discrete negations that goes from the drastic negation to the classical one.

Example 2. Let us consider some $\alpha \in L_n$ and consider the function N_α given by

$$N_\alpha(x) = \begin{cases} n & \text{if } x = 0, \\ \alpha - x & \text{if } 0 < x < \alpha, \\ 0 & \text{if } x \geq \alpha. \end{cases}$$

Then clearly N_α is a weak negation for all $\alpha \in L_n$. Moreover, when $\alpha = 0$ we obtain N_0 the drastic negation, whereas when $\alpha = n$ we obtain the classical negation $N_n(x) = n - x$.

4 Discrete t-subnorms and their associated negations

We want to discuss in this section the properties of the zero-region of a discrete t-subnorm in a similar way that it was done for t-subnorms defined on $[0, 1]$. To do this we begin with the following definition already considered in [13] for the $[0, 1]$-case.

Definition 4. *Given any discrete t-subnorm $T : L_n \times L_n \to L_n$, its* associated 0-function *is denoted by N_T and it is given by*

$$N_T(x) = \max\{z \in L_n \mid T(x, z) = 0\}.$$

Contrarily to what happens for t-norms, the associated 0-function of a t-subnorm does not need to be a discrete negation because $N_T(n) = \max\{z \in L_n \mid T(n, z) = 0\}$ could be different from 0 (when T is a proper t-subnorm n is not the neutral element of T). When N_T is in fact a discrete negation we will call it the *natural associated negation* of the t-subnorm T. Moreover, note that

1. When $n = 1$, the zero t-subnorm (with associated 0-function given by $N(x) = 1$ for $x \in \{0, 1\}$, which is not a negation), and the minimum t-norm (with the classical negation as natural associated negation) are the only t-subnorms on $L_1 = \{0, 1\}$.
2. When $n = 2$ there are exactly seven t-subnorms on L_2 that can be easily constructed, from which only two of them are t-norms. In any case, the only possibilities for their associated 0-functions are:
   - The constant function $N(x) = 2$ for all $x \in L_2 = \{0, 1, 2\}$.
   - $N(x) = \begin{cases} 2 & \text{if } x \in \{0, 1\}, \\ 1 & \text{if } x = 2. \end{cases}$
   - The classical one $N(x) = 2 - x$.
   - The drastic one $N(x) = \begin{cases} 2 & \text{if } x = 0, \\ 0 & \text{if } x \in \{1, 2\}. \end{cases}$.

   Clearly, only the last two cases are discrete negations.

In view of the above study, we will suppose $n \geq 3$ from now on. In the general case $n \geq 3$ we find also examples of t-subnorms, different from the zero t-subnorm, with associated 0-function that is not a negation as the following example shows.

Example 3. Let us consider $\alpha \in L_n$ and the function $T_\alpha : L_n^2 \to L_n$ given by

$$T_\alpha(x, y) = \max\{0, x + y - n - \alpha\} \qquad \text{for all} \quad x, y \in L_n.$$

Then T_α is always a smooth t-subnorm (see [20]) with $T_\alpha(n, n) = n - \alpha$. In particular, T_α is a proper t-subnorm if and only if $\alpha > 0$. Moreover, its associated 0-function is given by:

$$N_{T_\alpha}(x) = \max\{z \in L_n \mid T_\alpha(x, z) = 0\} = \max\{z \in L_n \mid x + z - n - \alpha \leq 0\}.$$

That is,

$$N_{T_\alpha}(x) = \min\{n, n + \alpha - x\} = \begin{cases} n & \text{if } x \leq \alpha, \\ n + \alpha - x & \text{if } x > \alpha. \end{cases}$$

Thus, it is clear that N_{T_α} is a discrete negation only when $\alpha = 0$ in which case T_α is in fact the Łukasiewicz t-norm. For all other cases $N_{T_\alpha}(n) = \alpha > 0$. This function N_{T_α} is depicted in Figure 1.

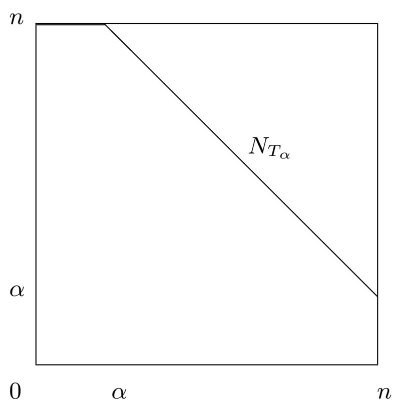

Fig. 1: The associated 0-function N_{T_α} of Example 3.

In the discrete case we have the following easy characterization of those t-subnorms for which their N_T is a negation.

Lemma 1. *Let $T : L_n^2 \to L_n$ be a discrete t-subnorm. The associated 0-function of T is a discrete negation if and only if $T(n, 1) = 1$.*

Proof. If N_T is a discrete negation then $N_T(n) = \max\{z \in L_n \mid T(x, n) = 0\} = 0$ and consequently $T(n, 1) > 0$. Since T is always under the minimum it must be $T(n, 1) = 1$.

Conversely, it is clear that when $T(n, 1) = 1$ then $N_T(n) = 0$ and N_T is a discrete negation. $\square$

Moreover, N_T turns out to be a weak negation and this property characterizes in fact those discrete negations that are associated to some t-subnorm with $T(n, 1) = 1$ (equivalently those that are associated to some t-norm) as follows.

Proposition 6. *Let $N : L_n \to L_n$ be a discrete negation. Then the following items are equivalent:*

i) There is some t-norm T such that $N = N_T$.
ii) There is some t-subnorm T with $T(n, 1) = 1$ such that $N = N_T$.
iii) N is a weak negation.

Proof. It is clear that $(i) \implies (ii)$.

In order to prove that $(ii) \implies (iii)$ suppose that there is some t-subnorm T with $T(n, 1) = 1$ such that $N = N_T$. In this case we have $N(x) = \max\{z \in L_n \mid T(x, z) = 0\}$ and consequently $T(x, N(x)) = 0$ and this directly implies that

$$N^2(x) = N(N(x)) = \max\{z \in L_n \mid T(N(x), z) = 0\} \geq x.$$

Finally, to prove $(iii) \implies (i)$ let N be a weak negation and consider the function given by

$$T(x, y) = \begin{cases} 0 & \text{if } y \leq N(x), \\ \min\{x, y\} & \text{if } y > N(x), \end{cases} \tag{2}$$

and let us prove that T is a t-norm[3], which clearly will have N as natural associated negation. Note that the function T given by Equation (2) is increasing in both variables and for all $x > 0$ we have $T(n, x) = \min\{n, x\} = x$, whereas $T(n, 0) = 0$ by definition. So, T has neutral element n. Since N is a weak negation, we have by Proposition 5 that $y \leq N(x)$ if and only if $x \leq N(y)$ and this implies that T is also commutative. Finally, it is an easy but tedious computation to prove the associativity and so T is a t-norm. □

In view of the previous proposition, when a discrete negation N is the associated negation of some t-subnorm it is also the associated negation of some t-norm. Nevertheless, there can be many more t-subnorms than t-norms having a specific weak negation as their associated negation (see for instance Proposition 9). However this is not the case when the associated negation is smooth, that is, when we consider the classical negation $N(x) = n - x$ as we prove in the following result, that can be viewed as the counterpart in the discrete case of Theorem 3.3 in [13].

Proposition 7. *Let $T : L_n^2 \to L_n$ be a discrete t-subnorm with natural associated negation $N_T(x) = n - x$. Then necessarily T is a t-norm.*

Proof. We only need to prove that $n \in L_n$ is the neutral element of T. Suppose on the contrary that there is some $x \in L_n$ such that $T(n, x) = x' < x$. Then

[3] For any weak negation N, the t-norm T given by Equation (2) is usually known as the *nilpotent minimum* with respect to N.

we have $n - x' > n - x$ and consequently it is $T(x, n - x') > 0$. Let us denote $y = T(x, n - x')$, then

$$0 = T(n - x', x') = T(n - x', T(x, n)) = T(T(n - x', x), n) = T(y, n),$$

which is a contradiction because by Lemma 1 it is $T(n, y) \geq T(n, 1) = 1$ for all $y > 0$. Consequently, it must be $T(n, x) = x$ for all $x \in L_n$ and T is a t-norm. $\square$

As a consequence of the previous proposition some known results about t-subnorms on $[0, 1]$, having strong associated negations (see [13, 14]), can be proved here for discrete t-subnorms. They have been compiled in the following proposition.

Proposition 8. *Let $T : L_n^2 \to L_n$ be a discrete t-subnorm with natural associated negation $N_T(x) = n - x$. The following items are equivalent:*

i) T is conditionally cancellative, i.e., for any $x, y, z \in L_n \setminus \{0\}$, $T(x, y) = T(x, z) > 0$ implies $y = z$.
ii) T is strictly increasing in its positive region, i.e., in $\{(x, y) \in (L_n \setminus \{0\})^2 \mid T(x, y) > 0\}$.
iii) T is smooth.
iv) T is the Łukasiewicz t-norm.

Proof. We will do only a sketch of the proof. It is obvious that $(i) \Longrightarrow (ii)$.

To prove $(ii) \Longrightarrow (iii)$ let us suppose $x, y \in L_n$ with $x < y$. We have then $n - y < n - x$ and so, $T(y, n - x) > 0$. Let us denote $z = T(y, n - x)$. Then a similar reasoning as in the [0,1]-case (see Theorem 2 in [14]) proves that it must be $T(y, n - z) = x$. That is, there is some $z \in L_n$ such that $T(y, z) = x$ and T satisfies the divisibility property which is equivalent to being T smooth (see Proposition 7.3.3 in [24]).

$(iii) \Longrightarrow (iv)$. Since T has associated negation $N_T(x) = n - x$ it must be a t-norm and the only smooth t-norm with this condition is the Łuckasiewicz t-norm.

Finally, $(iv) \Longrightarrow (i)$ is trivial. $\square$

Proposition 7 allows us to characterize all t-subnorms having natural associated negation in the special parametric family of discrete negations given in Example 2. We do it in the following theorem.

Proposition 9. *Let $T : L_n^2 \to L_n$ be a discrete t-subnorm. The following items hold:*

i) N_α is the natural associated negation of T if and only if T is an ordinal sum of a t-norm T' on $[0, \alpha]$ with the classical negation $N(x) = \alpha - x$ as associated negation and a t-subnorm T'' on $[\alpha, 1]$.
ii) If T is smooth then N_α is the natural associated negation of T if and only if T is an ordinal sum of the Łukasiewicz t-norm on $[0, \alpha]$ and a smooth t-subnorm T'' on $[\alpha, 1]$.

Proof. Let us prove first item (i). Suppose that N_α is the natural associated negation of T. Considering the restriction of T to the square $[0,\alpha]^2$, we clearly obtain a t-subnorm, $T' = T_{/[0,\alpha]^2}$, on the finite chain $[0,\alpha]$ with natural associated negation given by $N(x) = \alpha - x$ for all $x \in [0,\alpha]$. Applying Proposition 7 we deduce that T' must be a t-norm and consequently we have in particular that $T(\alpha,\alpha) = \alpha$. In this case, it is well known that T must be an ordinal sum of the t-norm T' on $[0,\alpha]$ and a t-subnorm T'' on $[\alpha,1]$.

Conversely, it is clear that any t-subnorm given by an ordinal sum of a t-norm T' on $[0,\alpha]$ with associated negation $N(x) = \alpha - x$ and a t-subnorm T'' on $[\alpha,1]$, has N_α as natural associated negation.

Finally, note that item (ii) is an immediate consequence of the previous item and Proposition 8.

□

The structure of the t-subnorms characterized in Proposition 9 can be viewed in Figure 2.

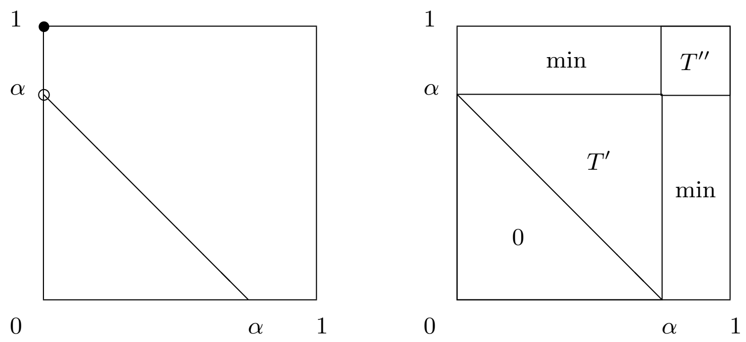

Fig. 2: Negations N_α (left) of the parametric family given in Example 2 and the structure of the t-subnorms (right) having N_α as natural associated negation characterized in Proposition 9-(i).

5 Conclusions and future work

In this paper, an in-depth study on the natural associated negations of discrete t-subnorms has been performed. First of all, the equivalence of weak negations and symmetrical negations has been proved in the discrete setting in contrast with what happens in $[0,1]$ where this equivalence does only hold when the fuzzy negation is left-continuous. After that, the concept of 0-function of a discrete t-subnorm has been introduced, adapting it from the $[0,1]$-case, and the cases when this function is in fact a discrete negation have been characterized.

From this study, several results concerning the relationship between discrete t-subnorms and discrete t-norms according to the properties of their natural associated negation have been presented. Of particular importance is Proposition 6 which proves that weak negations are the only discrete negations which are the natural associated negations of a discrete t-norm.

As future work, we want to analyse this topic from the other perspective, that is, if we consider a fixed weak negation N, which t-norms T can be considered in order to get a new t-norm T' such that $N_{T'} = N$ and $T'(x,y) = T(x,y)$ for all $y > N(x)$? Equivalently, characterize for which t-norms T the operator given by

$$T'(x,y) = \begin{cases} 0 & \text{if } y \leq N(x), \\ T(x,y) & \text{if } y > N(x), \end{cases}$$

is a t-norm. This problem was already tackled in [4] in the $[0,1]$-framework but it has not been studied yet in the discrete setting.

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