Natural negations associated to discrete t-subnorms

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Outline

Motivation: From $[0,1]$ to the discrete chain

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In many applications, the range of computations and reasoning must be reduced to a finite set of possible values, often qualitative leading to the so-called **fuzzy linguistic approach**.

In this case, the qualitative terms used by experts are usually represented via linguistic variables instead of numerical values, often interpreted to take values on totally ordered scales such as:

$$L = \{\text{Extremely Bad}, \text{Very Bad}, \text{Bad}, \text{Fair}, \text{Good}, \text{Very Good}, \text{Extremely Good}\},$$

which can be all represented by the finite chain $L_n = \{0, 1, \ldots, n\}$.

Discrete aggregation functions

Consequently, many researchers have focused their efforts to study operations defined on L_n, or **discrete operations**.

Among the whole set of discrete operations, **discrete aggregation functions** stand out due to the necessity to merge some data into a representative output. They play an important role in different applications such as:

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Among the whole set of discrete operations, **discrete aggregation functions** stand out due to the necessity to merge some data into a representative output. They play an important role in different applications such as:

- decision making,
- image processing and pattern recognition,
- approximate reasoning.

Discrete aggregation functions

Many families of discrete aggregation functions have already been studied or even characterized. For instance,

- smooth t-norms and t-conorms,
- smooth t-subnorms,
- weighted ordinal means,
- uninorms in $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ and nullnorms,
- discrete idempotent uninorms,
- uninorms and nullnorms without the commutative property,
- copulas and quasi-copulas.

Discrete t-subnorms

In this paper, we want to focus on **discrete t-subnorms**.

Definition

Let $T : L_n^2 \to L_n$ be a binary operation on L_n. Then T is said to be a *t-subnorm* when T is associative, commutative, non-decreasing in each variable and such that $T(x, y) \leq \min\{x, y\}$ for all $x, y \in L_n$.

Discrete t-subnorms

These operations are important due to the following reasons:

1. These operations generalize the well-known family of discrete t-norms.
2. They are a particular case of an order topological semigroup.
3. T-subnorms on $[0, 1]$ play a key role in the ordinal sum based construction of left-continuous t-norms.

Discrete t-subnorms

Focusing on t-subnorms in $[0, 1]$, recently it has been studied that the properties of the associated natural negations of the t-subnorms are decisive when studying their relation with t-norms and the fulfilment of properties such as:

- Archimedeanness,
- conditional cancellativity,
- left-continuity,
- nilpotent elements.

Goal

To perform a similar study for discrete t-subnorms proving several insights about the structure and additional properties of these operators.

Preliminaries

Smoothness Property

Definition

- A function $f : L_n \to L_n$ is said to be *smooth* if it satisfies $|\, f(x) - f(x-1)\,| \leq 1$ for all $x \in L_n$ with $x \geq 1$.
- A binary operation F on L_n is said to be *smooth* when each one of its vertical and horizontal sections are smooth.

It is a discrete counterpart of continuity on $[0, 1]$, because it is equivalent to the divisibility property.

Discrete t-norms

Definition

Let $T : L_n^2 \to L_n$ be a binary operation on L_n. Then T is said to be a *t-norm* when T is associative, commutative, non-decreasing in each variable and $T(n, x) = x$ for all $x \in L_n$.

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Any t-norm on L_n is also a t-subnorm but not vice versa.

Discrete t-norms: Ordinal sums

Proposition

Let $0 = a_0 < a_1 < \ldots < a_{m-1} < a_m = n$ *be* $m + 1$ *elements in* L_n *and let* T_i *be a t-norm on the chain* $[a_{i-1}, a_i]$ *for all* $i = 1, \ldots, m$*. Then the binary operation on* L_n *given by*

$$T(x, y) = \begin{cases} T_i(x, y) & \textit{if there is an } i \textit{ such that } a_{i-1} \leq x, y \leq a_i, \\ \min\{x, y\} & \textit{otherwise,} \end{cases}$$

is always a t-norm on L_n *usually called the ordinal sum of t-norms* $T_1, \ldots, T_m$*.*

Discrete t-norms: Smooth Ordinal sums

Proposition

There exists one and only one Archimedean smooth t-norm on L_n which is given by

$$T(x, y) = \max\{0, x + y - n\}$$

and it is usually known as the Łukasiewicz t-norm.

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Proposition

A t-norm T on L_n is smooth if and only if there exists a natural number m with $1 \leq m \leq n$ and a subset J of L_n,

$$J = \{0 = a_0 < a_1 < \ldots < a_{m-1} < a_m = n\}$$

such that T is given by

$$T(x, y) = \begin{cases} \max\{a_k, x + y - a_{k+1}\} & \textit{if there is } a_k \in J \textit{ with } a_k \leq x, y \leq a_{k+1}, \\ \min\{x, y\} & \textit{otherwise.} \end{cases}$$

Some properties on discrete negations

Let us start with a well-known result.

Proposition

The only smooth (equivalently strong or strictly decreasing) negation on L_n is the classical negation given by

$$N(x) = n - x \quad \textit{for all} \quad x \in L_n.$$

However, in the non-smooth case we can find many other possibilities for discrete negations.

Weak and symmetrical negations

Definition

Let $N : L_n \to L_n$ be a discrete negation.

- N is said to be a *weak negation* when $x \leq N^2(x)$ for all $x \in L_n$.
- N is said to be *symmetrical* when the set

$$F_N = \{(n, 0)\} \cup \{(x, y) \in L_n^2 \mid N(x + 1) \leq y \leq N(x)\}$$

is symmetrical, that is, $(x, y) \in F_N$ if and only if $(y, x) \in F_N$.

First difference with respect to the $[0,1]$-case

In the case of the interval $[0,1]$, weak and symmetrical negations do not coincide in general and they coincide only when the negation N is left-continuous, but. . .

Proposition

Let $N : L_n \to L_n$ be a discrete negation. The following items are equivalent:

i) *N is symmetrical.*

ii) *N is a weak negation.*

iii) *For all $(x, y) \in L_n^2$ it holds that:*

$$y \leq N(x) \iff x \leq N(y).$$

Example

Example

Let us consider some $\alpha \in L_n$ and consider the function N_α given by

$$N_\alpha(x) = \begin{cases} n & \text{if } x = 0, \\ \alpha - x & \text{if } 0 < x < \alpha, \\ 0 & \text{if } x \geq \alpha. \end{cases}$$

Then clearly N_α is a weak negation for all $\alpha \in L_n$. Moreover, when $\alpha = 0$ we obtain N_0 the drastic negation, whereas when $\alpha = n$ we obtain the classical negation $N_n(x) = n - x$.

Discrete t-subnorms and their associated negations

Zero-function

Definition

Given any discrete t-subnorm $T : L_n \times L_n \to L_n$, its *associated 0-function* is denoted by N_T and it is given by

$$N_T(x) = \max\{z \in L_n \mid T(x, z) = 0\}.$$

Zero-function

Definition

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$$N_T(x) = \max\{z \in L_n \mid T(x, z) = 0\}.$$

However, $N_T(n) \neq 0$ in general. Whenever $N_T(n) = 0$, we will call it the *natural associated negation* of the t-subnorm T.

Some examples

Example

1. When $n = 1$, the zero t-subnorm (with associated 0-function given by $N(x) = 1$ for $x \in \{0, 1\}$, which is not a negation), and the minimum t-norm (with the classical negation as natural associated negation) are the only t-subnorms on $L_1 = \{0, 1\}$.
2. When $n = 2$ there are exactly seven t-subnorms on L_2, from which only two of them are t-norms. In any case, the only possibilities for their associated 0-functions are:
   - The constant function $N(x) = 2$ for all $x \in L_2 = \{0, 1, 2\}$.
   - $N(x) = \begin{cases} 2 & \text{if } x \in \{0, 1\}, \\ 1 & \text{if } x = 2. \end{cases}$
   - The classical one $N(x) = 2 - x$.
   - The drastic one $N(x) = \begin{cases} 2 & \text{if } x = 0, \\ 0 & \text{if } x \in \{1, 2\}. \end{cases}$.

   Clearly, only the last two cases are discrete negations.

t-subnorms with associated natural negation

Lemma

Let $T : L_n^2 \to L_n$ be a discrete t-subnorm. The associated 0-function of T is a discrete negation if and only if $T(n, 1) = 1$.

Moreover, we can go one step further.

t-subnorms with associated natural negation

Proposition

Let $N : L_n \to L_n$ be a discrete negation. Then the following items are equivalent:

i) *There is some t-norm T such that $N = N_T$.*

ii) *There is some t-subnorm T with $T(n, 1) = 1$ such that $N = N_T$.*

iii) *N is a weak negation.*

t-subnorms with associated natural negation

Proposition

Let $T : L_n^2 \to L_n$ be a discrete t-subnorm with natural associated negation $N_T(x) = n - x$. Then necessarily T is a t-norm.

t-subnorms with associated natural negation

Proposition

Let $T : L_n^2 \to L_n$ be a discrete t-subnorm with natural associated negation $N_T(x) = n - x$. The following items are equivalent:

i) *T is conditionally cancellative, i.e., for any $x, y, z \in L_n \setminus \{0\}$, $T(x, y) = T(x, z) > 0$ implies $y = z$.*

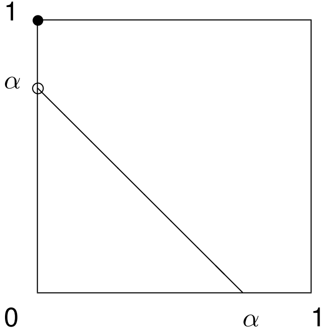
ii) *T is strictly increasing in its positive region, i.e., in $\{(x, y) \in (L_n \setminus \{0\})^2 \mid T(x, y) > 0\}$.*

iii) *T is smooth.*

iv) *T is the Łukasiewicz t-norm.*

t-subnorms with associated natural negation

However, we can characterize not only those t-subnorms which have the classical negation as associated natural negation.



t-subnorms with associated natural negation

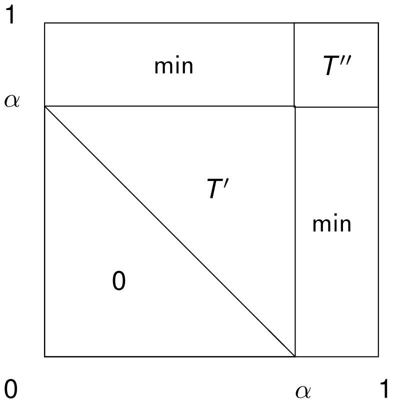
Proposition

Let $T : L_n^2 \to L_n$ be a discrete t-subnorm. The following items hold:

i) *N_α is the natural associated negation of T if and only if T is an ordinal sum of a t-norm T' on $[0, \alpha]$ with the classical negation $N(x) = \alpha - x$ as associated negation and a t-subnorm T'' on $[\alpha, 1]$.*

ii) *If T is smooth then N_α is the natural associated negation of T if and only if T is an ordinal sum of the Łukasiewicz t-norm on $[0, \alpha]$ and a smooth t-subnorm T'' on $[\alpha, 1]$.*

t-subnorms with associated natural negation



t-subnorms with associated natural negation

When a discrete negation N is the associated negation of some t-subnorm it is also the associated negation of some t-norm. Nevertheless, there can be many more t-subnorms than t-norms having a specific weak negation as their associated negation.

Conclusions and Future Work

Conclusions

In this paper, we have:

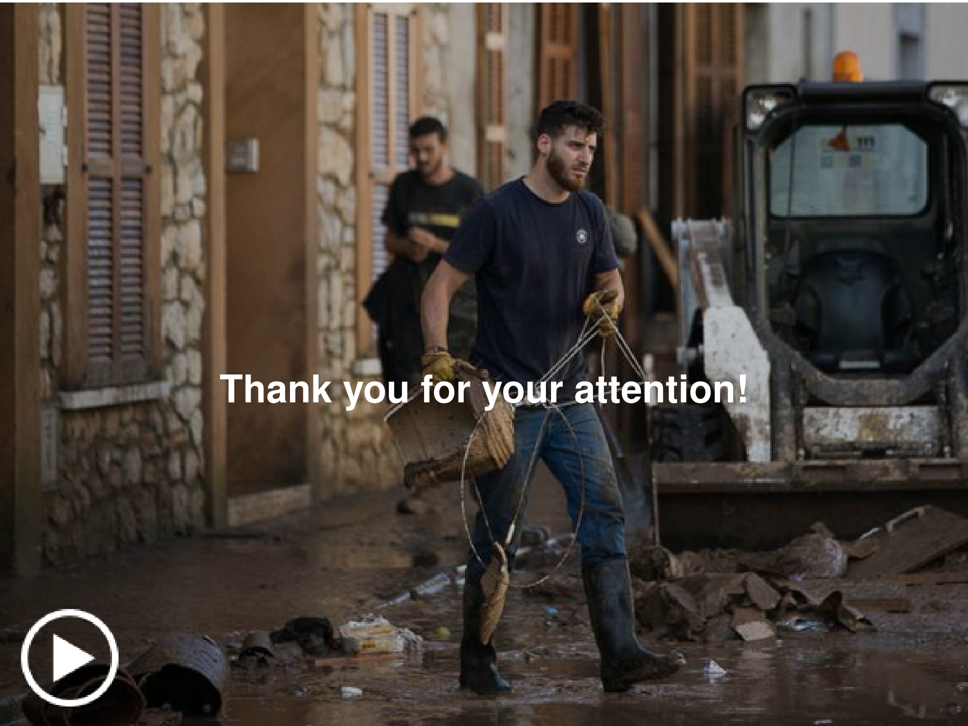
- studied the natural associated negations of discrete t-subnorms,
- proved the equivalence of weak negations and symmetrical negations in the discrete setting,
- proved several results concerning the relationship between discrete t-subnorms and discrete t-norms according to the properties of their natural associated negation.

Future Work

As future work, we want to analyse this topic from the other perspective, that is, if we consider a fixed weak negation N, which t-norms T can be considered in order to get a new t-norm T' such that $N_{T'} = N$ and $T'(x, y) = T(x, y)$ for all $y > N(x)$? Equivalently, characterize for which t-norms T the operator given by

$$T'(x,y) = \begin{cases} 0 & \text{if } y \leq N(x), \\ T(x,y) & \text{if } y > N(x), \end{cases}$$

is a t-norm.


Thank you for your attention!