Tidying up the mess of classes of fuzzy implication functions

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Outline

1. Plenary talk preparation
2. Fuzzy implications: some basics
3. The mess of classes of fuzzy implication functions
4. Good and not-so-good practices
5. Conclusions and looking to the future

Plenary talk preparation

Plenary talk preparation

When the organizers invited me to give this plenary talk, I asked myself the next questions:

1. What can I talk about? **(TOPIC)**
2. Which type of plenary talk I would like to give? **(FORMAT)**
3. Which goals do I want to achieve? **(GOALS)**

What can I talk about?

Two options:

(a) Chess

(b) Fuzzy implications

What can I talk about?

Which type of plenary talk I would like to give?

Several plenary talks focused on fuzzy implication functions in recent years:

Which type of plenary talk I would like to give?

Several plenary talks focused on fuzzy implication functions in recent years:

- Plenary talks focused on basics and state of the art:
  - Michał Baczyński, “Fuzzy Implication Functions: Recent Advances”, **EUSFLAT 2011**, Aix-les-Bains.
  - Michał Baczyński, “Functional Equations Involving Fuzzy Implications and Their Applications in Approximate Reasoning”, **AGOP 2013**, Pamplona.

Which type of plenary talk I would like to give?

Several plenary talks focused on fuzzy implication functions in recent years:

- Plenary talks focused on some novel research line:
  - Balasubramaniam Jayaram, “Fuzzy implications: some algebraic perspectives”, **AGOP 2015**, Katowice.

Which type of plenary talk I would like to give?

In my case,

- Plenary “position” talk.
  - My **subjective and personal** opinion about the current state of the research on fuzzy implications.

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Which goals do I want to achieve?

1. Open a discussion about the future of the theoretical research on fuzzy implications.

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3. Provide some examples of good practices and motivations.

Which goals do I want to achieve?

1. Open a discussion about the future of the theoretical research on fuzzy implications.
2. Present the existing mess on families of fuzzy implications.
3. Provide some examples of good practices and motivations.
4. Expose some not-so-good practices.

Fuzzy implications: some basics

Fuzzy implications

Let us start with the definition.

Definition (Kitainic, 1993)

A binary operation $I : [0,1]^2 \to [0,1]$ is said to be a *fuzzy implication* if it satisfies:

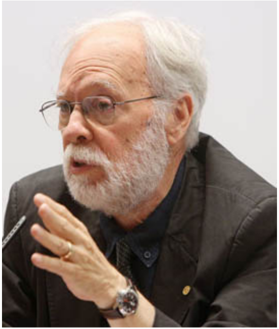
(I1) $I(x,z) \geq I(y,z)$ when $x \leq y$, for all $z \in [0,1]$.

(I2) $I(x,y) \leq I(x,z)$ when $y \leq z$, for all $x \in [0,1]$.

(I3) $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.

Discussion about the name

Why **fuzzy** implications?



Discussion about the name

Why **fuzzy** implications?

- It generalizes the classical implication to fuzzy logic.
- It is the usual name to denote implications in fuzzy logic from the seventies.

Discussion about the name

We should distinguish between:

- implications on fuzzy propositions $\Rightarrow$ **fuzzy implications**,
- functions that can be used to build fuzzy implications $\Rightarrow$ **fuzzy implication functions**.

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We should distinguish between:

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- functions that can be used to build fuzzy implications $\Rightarrow$ **fuzzy implication functions**.

When we consider a binary operator $I : [0,1] \times [0,1] \to [0,1]$ such that

$$(A \to B)(x,y) = I(A(x), B(y))$$

once selected the implication function I, the value $(A \to B)(x,y)$ depends only on the values taken by A and B at the points x and y. However,

Discussion about the name

However,

Definition (Massanet, Mayor, Mesiar, Torrens, 2013)

An operation $\to$ defined from $[0,1]^X \times [0,1]^Y$ to $[0,1]^{X\times Y}$, where for all $(A,B) \in [0,1]^X \times [0,1]^Y$, it returns $A \to B \in [0,1]^{X\times Y}$ is a *fuzzy implication* if the following conditions hold:

Im1) If $A \leq A'$ then $A \to B \geq A' \to B$ for all $B \in [0,1]^Y$, i.e., $\to$ is decreasing in the first variable.

Im2) If $B \leq B'$ then $A \to B \leq A \to B'$ for all $A \in [0,1]^X$, i.e., $\to$ is increasing in the second variable.

Im3) If $A \in \{0_X, 1_X\}$ and $B \in \{0_Y, 1_Y\}$ then

$$(A \to B) = \begin{cases} 0_{X\times Y} & \text{if } A = 1_X \text{ and } B = 0_Y, \\ 1_{X\times Y} & \text{otherwise,} \end{cases}$$

i.e., $\to$ extends the crisp implication.

Discussion about the name

Proposition (Massanet, Mayor, Mesiar, Torrens, 2013)

A fuzzy implication $\to$ is functionally expressible if, and only if, there exists a function $I : [0,1] \times [0,1] \longrightarrow [0,1]$ such that $(A \to B)(x,y) = I(A(x), B(y))$ for all $A \in [0,1]^X$, $B \in [0,1]^Y$, $(x,y) \in X \times Y$ satisfying:

i) *If $a \leq a'$, then $I(a,b) \geq I(a',b)$ for all $b \in [0,1]$, i.e., I is non-increasing in the first variable.*

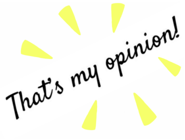
ii) *If $b \leq b'$, then $I(a,b) \leq I(a,b')$ for all $a \in [0,1]$, i.e., I is non-decreasing in the second variable.*

iii) *$I(0,0) = I(1,1) = I(0,1) = 1$ and $I(1,0) = 0$ (boundary conditions).*

Discussion about the name

Thus, only functionally-expressible fuzzy implications are “fuzzy implications” in the sense of the first definition. Consequently, a better name is **fuzzy implication functions**.

> *S. Massanet, G. Mayor, R. Mesiar, J. Torrens: On fuzzy implications: An axiomatic approach. Int. J. Approx. Reasoning* ***54****(9): 1471-1482 (2013)*



The game-changing moment

Fuzzy implication functions have been studied from the beginnings of fuzzy logic and fuzzy sets. Some important old contributions can be found in:

P. Smets, P. Magrez, Implication in fuzzy logic, International Journal of Approximate Reasoning 1 (1987) 327-347.

J.C. Fodor, M. Roubens: Fuzzy preference modelling and multicriteria decision support. Kluwer, Dordrecht (1994)

G.J. Klir, B. Yuan: Fuzzy sets and fuzzy logic. Theory and applications. Prentice Hall, New Jersey (1995)

The game-changing moment

However, the game-changing publications were:

IEEE TRANSACTIONS ON FUZZY SYSTEMS, VOL. 15, NO. 6, DECEMBER 2007 1107

A Survey on Fuzzy Implication Functions

M. Mas, M. Monserrat, J. Torrens, and E. Trillas

***Abstract*—One of the key operations in fuzzy logic and approximate reasoning is the fuzzy implication, which is usually performed by a binary operator I, called an implication function or, simply, an implication. Many fuzzy rule based systems do their inference processes through these operators that also take charge of the propagation of uncertainty in fuzzy reasonings. Moreover, they have proved to be useful also in other fields like composition of fuzzy relations, fuzzy relational equations, fuzzy mathematical morphology, and image processing. This paper aims to present an overview on fuzzy implication functions that usually are constructed from t-norms and t-conorms but also from other kinds of aggregation operators. The four most usual ways to define these implications are recalled and their characteristic properties stated, not only in the case of [0,1] but also in the discrete case.**

***Index Terms*—Aggregation function, discrete implication, functional equation, implication function, t-conorm, t-norm, uninorm.**

I. Introduction

ONE of the most important problems in fuzzy logic is the management of fuzzy conditionals of the type "If p, then q" with p and q fuzzy statements. One method commonly used to do this management is through functions $I : [0,1] \times [0,1] \to [0,1]$ in such a way that the truth value of the conditional is functionally stated from the truth values of the initial propositions p and q. These functions I are called *(fuzzy) implication functions* or simply *(fuzzy) implications*. Traditionally, it is accepted that any fuzzy concept must generalize the corresponding crisp concept and, consequently, it is usually required of an implication function that $I(0,0) = I(0,1) = I(1,1) = 1$ and $I(1,0) = 0$; that is, its restriction to $\{0,1\}^2$ must coincide with the classical material implication $p \to q \equiv \neg p \vee q$.[1] Of course, several other conditions are also required of implication functions depending on the context, generally taken from properties of the classical implication.

Since conjunctions, disjunctions, and negations are usually performed by t-norms, t-conorms, and strong negations, in fuzzy set theory as much as in fuzzy logic, the majority of the known implication functions are directly derived from these operators. The four most usual ways to define these implication functions are the following (see, for instance, [38], [23], and [67]).

1) *S-implications* defined by

$$I(x,y) = S(N(x),y), \quad x,y \in [0,1] \tag{1}$$

where S is a t-conorm and N is a strong negation. They appear as an immediate generalization of the classical boolean implication $p \to q \equiv \neg p \vee q$.

2) *R-implications* defined by

$$I(x,y) = \sup\{z \in [0,1] \mid T(x,z) \le y\}, \quad x,y \in [0,1] \tag{2}$$

where T is a left-continuous t-norm.[2] They come from residuated lattices based on the residuation property that in the case of t-norms can be written as

$$T(x,y) \le z \Longleftrightarrow I(x,z) \ge y \quad \text{for all } x,y,z \in [0,1]. \tag{3}$$

3) *QL-implications* defined by

$$I(x,y) = S(N(x),T(x,y)), \quad x,y \in [0,1] \tag{4}$$

where T is a t-norm, S is a t-conorm, and N is a strong negation. They come from quantum mechanic logic.

4) *D-implications*, that are the contraposition with respect to N of QL-implications and are given by

$$I(x,y) = S(T(N(x),N(y)),y), \quad x,y \in [0,1] \tag{5}$$

where T is a t-norm, S is a t-conorm, and N is a strong negation. They are known under this name since they come from the Dishkant arrow $p \to q \equiv q \vee (\neg p \wedge \neg q)$ in orthomodular lattices (see [67]).

Of course, all these expressions for implications are equivalent in any Boolean algebra and consequently in classical logic. However, in fuzzy logic these four definitions yield to distinct classes of fuzzy implications. Thus, the following question naturally arises: Why so many different models to perform this kind of operation? The main reason is because they are used to represent imprecise knowledge. Note that any "if … then" rule in fuzzy systems is interpreted through one of these implication functions. So, depending on the context, and on the proper rule and its behavior, different implications can be adequate in any case. Moreover, since implications are used not only to represent conditional statements but also, and mainly, to perform inferences (see Section I-A and -D, respectively, devoted to approximate reasoning and fuzzy control), different kinds of implementations are useful. Specifically, the two main classical inference rules are *modus ponens* and *modus tollens*, used to per-

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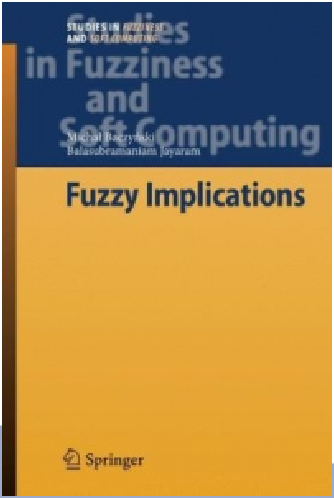
[1]However, in the case of implication functions, because they can be generalized from the case of lattices, where it is only required that implications satisfy $x \cdot (x \to y) \le y$—that is, to be conditionals—other approximations are accepted and commonly used, especially in fuzzy control, like the proper conjunction.

[2]The residuation property given by (3) holds for t-norms T if and only if T is left-continuous, and for this reason many authors consider R-implications only for left-continuous t-norms. However, there are also some authors that consider them for any t-norm and some others only for continuous t-norms.

*M. Mas, M. Monserrat, J. Torrens, E. Trillas: A Survey on Fuzzy Implication Functions. IEEE Trans. Fuzzy Systems **15**(6): 1107-1121 (2007)*

The game-changing moment

However, the game-changing publications were:



M. Baczyński, B. Jayaram: Fuzzy Implications. Studies in Fuzziness and Soft Computing ***231****, Springer, 2008.*

In the last decade...

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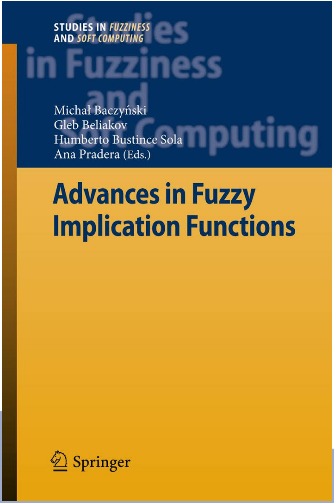

Advances in Fuzzy Implication Functions. (Eds.) M. Baczyński, G. Beliakov, H. Bustince, A. Pradera. Studies in Fuzziness and Soft Computing ***300***, *Springer, 2013.*

In the last decade...

More surveys and books entirely devoted to fuzzy implication functions:

12. Fuzzy Implications: Past, Present, and Future

Michał Baczynski, Balasubramaniam Jayaram, Sebastia Massanet, Joan Torrens

Fuzzy implications are a generalization of the classical two-valued implication to the multi-valued setting. They play a very important role both in the theory and applications, as can be seen from their use in, among others, multivalued mathematical logic, approximate reasoning, fuzzy control, image processing, and data analysis. The goal of this chapter is to present the evolution of fuzzy implications from their beginnings to the current days. From the theoretical point of view, we present the basic facts, as well as the main topics and lines of research around fuzzy implications. We also devote a specific section to state and recall a list of main application fields where fuzzy implications are employed, as well as another one to the main open problems on the topic.

Part B | 12

Fuzzy logic connectives play a fundamental role in the theory of fuzzy sets and fuzzy logic. The basic fuzzy connectives that perform the role of generalized *And*, *Or*, and *Not* are t-norms, t-conorms, and negations, respectively, whereas fuzzy conditionals are usually managed through fuzzy implications. Fuzzy implications play a very important role both in theory and applications, as can be seen from their use in, among others, multivalued mathematical logic, approximate reasoning, fuzzy control, image processing, and data analysis. Thus, it is hardly surprising that many researchers have devoted their efforts to the study of implication functions. This interest has become more evident in the last decade when many works have appeared and have led to some surveys [12.1, 2] and even some research monographs entirely devoted to this topic [12.3, 4]. Thus, most of the known results and applications of fuzzy implications until the publication date were collected in [12.3], and very recently the edited volume [12.4] has been published complimenting the earlier monograph with the most recent lines of investigation on fuzzy implications.

In this regard, we have decided to devote this chapter, as the title suggests, to present the evolution of fuzzy implications from their beginnings to the present time. The idea is not to focus on a list of results already collected in other works, but unraveling the relations and highlighting the importance in the development and progress that fuzzy implications have experienced along the time. From the theoretical point of view we present the basic facts, as well as the main topics and lines of research around fuzzy implications, recalling in most of the cases where the corresponding results can be found, instead of listing them. Of course, we also devote a specific section to state and recall a list of the main application fields where fuzzy implications are employed. A final section looks ahead to the future by listing some of the main open-problem-solutions of which are certain to enrich the existing literature on the topic.

M. Baczyński, B. Jayaram, S. Massanet, J. Torrens: Fuzzy Implications: Past, Present, and Future. Handbook of Computational Intelligence, 183-202 (2015).

Consequences

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- Large development in the theory and applications ✓

Consequences

- Large development in the theory and applications $\checkmark$
- New applications $\checkmark$

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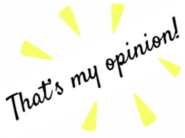
- Too many untouched open problems ✗
- Lack of clear goals ✗



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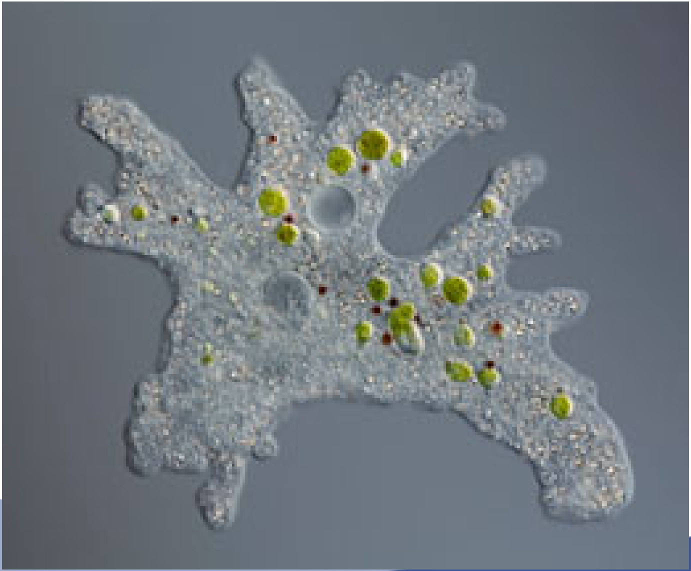

Where does the research on fuzzy implication functions go?

The mess of classes of fuzzy implication functions

Most researched theoretical lines

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4. The analysis of the additional properties fulfilled by the existing classes of fuzzy implication functions.

These lines of research are a direct consequence of the definition of these operators.

Definition of fuzzy implication functions

Definition

A binary operation $I : [0,1]^2 \to [0,1]$ is said to be a *fuzzy implication function* if it satisfies:

(I1) $I(x,z) \geq I(y,z)$ when $x \leq y$, for all $z \in [0,1]$.

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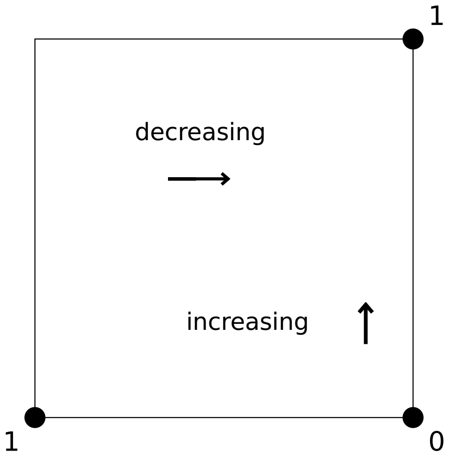
Flexibility on the definition



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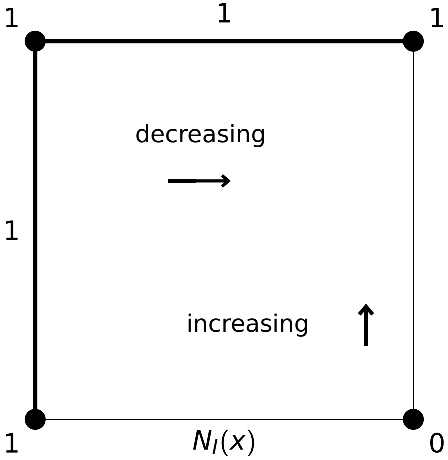

Classes of fuzzy implication functions

This flexibility allows the existence of an infinite number of classes of fuzzy implication functions with different structures, performances on the applications and additional properties.

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2. Using unary monotone functions.

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2. Using unary monotone functions.
3. Constructing fuzzy implication functions from other given fuzzy implication functions.

Strategy 1: Combinations of fuzzy logical operators

(S,N)-implications

Definition (Alsina, Trillas, 2003)

Let S be a t-conorm and N a fuzzy negation. Then the binary operator $I_{S,N} : [0,1]^2 \to [0,1]$ given by

$$I_{S,N}(x, y) = S(N(x), y), \quad \text{for all } x, y \in [0,1]$$

is called the (S, N)-implication derived from S and N.

They generalize the classical material implication $\neg p \vee q$.

Strategy 1: Combinations of fuzzy logical operators

Generalizations of (S,N)-implications

Several generalizations have been proposed by changing S by:

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   M. Baczynski, B. Jayaram: (U, N)-implications and their characterizations. Fuzzy Sets and Systems ***160****(14): 2049-2062 (2009)*

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2. Co-copula.

   R. R. Yager: Modeling holistic fuzzy implication using co-copulas. FO & DM ***5****(3): 207-226 (2006)*

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3. TS-functions given by

$$F(x,y) = f^{-1}((1-\lambda)f(T(x,y)) + \lambda f(S(x,y))), \quad \text{for all } x,y \in [0,1].$$

   H. Bustince, J. Fernandez, A. Pradera, G. Beliakov: On (TS,N)-fuzzy implications. Proc. AGOP 2011, pp. 93-98, 2011.

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   Z.-B. Li, Y. Su and H.-W. Liu: Generalization of (U,N)-implications. Int. Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, Vol. ***23****, No. 03, pp. 367-377 (2015)*

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5. Commutative semi-uninorms and pseudo-uninorms.

   Z.-B. Li and H.-W. Liu: Generalizations of (U, N)-implications derived from commutative semi-uninorms and pseudo-uninorms . Journal of Intelligent & Fuzzy Systems, vol. ***29****, no. 5, pp. 2177-2184, 2015*

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6. Any aggregation function.

   Y. Ouyang: On fuzzy implications determined by aggregation operators. Information Sciences, Vol. ***193****, 153-162 (2012)*

Strategy 1: Combinations of fuzzy logical operators

Generalizations of (S,N)-implications

Several generalizations have been proposed by changing S by:

7. Grouping function $\Rightarrow$ (G, N)-implications.

   G.P. Pereira, B. Bedregal, R.H.N. Santiago: On (G,N)-implications derived from grouping functions. Information Sciences, Vol. ***279****, pp. 1-17, 2014.*

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8. Many others...

Strategy 1: Combinations of fuzzy logical operators

R-implications

Definition (Pedrycz, 1982; Miyakoshi, Shimbo, 1985)

Let T be a t-norm. Then the binary operator $I_T : [0,1]^2 \to [0,1]$ given by

$$I_T(x,y) = \sup\{z \in [0,1] \mid T(x,z) \le y\}, \quad \text{for all } x,y \in [0,1]$$

is called the R-implication derived from T.

They come from the residuated lattice theory and they satisfy the *residuation property*:

$$T(x,y) \le z \Leftrightarrow I_T(x,z) \ge y.$$

Strategy 1: Combinations of fuzzy logical operators

Generalizations of R-implications

Several generalizations have been proposed by changing T by:

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2. Copula, quasi-copula and semi-copula.

   F. Durante, E.P. Klement, R. Mesiar, C. Sempi: Conjunctors and their residual implicators: characterizations and construction methods. Mediterranean Journal of Mathematics, vol. ***4****, no. 3, pp. 343-356, 2007.*

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3. Any aggregation function.

   Y. Ouyang: On fuzzy implications determined by aggregation operators. Information Sciences, Vol. ***193****, 153-162 (2012)*

Strategy 1: Combinations of fuzzy logical operators

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Several generalizations have been proposed by changing T by:

4. Overlap functions $\Rightarrow$ R_O-implications.

   *B. Bedregal and G.P. Dimuro: On residual implications derived from overlap functions. Information Sciences Vol. **312**, pp. 78-88, 2015.*

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5. Many others...

Strategy 1: Combinations of fuzzy logical operators

Definition (Trillas, Valverde, 1981)

Let T be a t-norm, S a t-conorm and N a fuzzy negation. Then:

- the binary operator $I_{T,S,N} : [0,1]^2 \to [0,1]$ given by

$$I_{T,S,N}(x,y) = S(N(x), T(x,y)), \quad \text{for all } x, y \in [0,1]$$

  is called the *QL*-implication derived from T, S and N,

- the binary operator $I^{T,S,N} : [0,1]^2 \to [0,1]$ given by

$$I^{T,S,N}(x,y) = S(y, T(N(x), N(y))), \quad \text{for all } x, y \in [0,1]$$

  is called the *D*-implication derived from T, S and N.

They come from Quantum logic and the Dishkant implication, respectively. They are very related.

Strategy 1: Combinations of fuzzy logical operators

Generalizations of QL-implications and D-implications

Several generalizations have been proposed by changing T and S by:

Strategy 1: Combinations of fuzzy logical operators

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Several generalizations have been proposed by changing T and S by:

1. Conjunctive and disjunctive uninorms.

   M. Mas, M. Monserrat, J. Torrens: Two types of implications derived from uninorms. Fuzzy Sets and Systems ***158****(23): 2612-2626 (2007).*

Strategy 1: Combinations of fuzzy logical operators

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2. Overlap and Grouping operators.

   G.P. Dimuro, B. Bedregal, H. Bustince, A. Jurio, M. Baczyński, K. Mis: QL-operations and QL-implication functions constructed from tuples (O,G,N) and the generation of fuzzy subsethood and entropy measures. Int. J. Approx. Reasoning ***82****: 170-192 (2017).*

Strategy 1: Combinations of fuzzy logical operators

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3. Others.

Strategy 1: Combinations of fuzzy logical operators

Classes related to probability theory

Definition (Grzegorzewski, 2011,2013)

Let C be a copula. Then:

- the function $I_C : [0,1]^2 \to [0,1]$ given by

$$I_C(x,y) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{C(x,y)}{x} & \text{if } x > 0, \end{cases}$$

  is called the *probabilistic implication* based on the copula C,

- the function $I_C^* : [0,1]^2 \to [0,1]$ given by

$$I_C^*(x,y) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{x+y-1+C(1-x,1-y)}{x} & \text{if } x > 0, \end{cases}$$

  is called the *survival implication* based on the copula C.

Strategy 1: Combinations of fuzzy logical operators

Classes related to probability theory

Definition (Grzegorzewski, 2011,2013)

Let C be a copula. Then:

- the function $\tilde{I}_C : [0,1]^2 \to [0,1]$ given by

$$\tilde{I}_C(x,y) = C(x,y) - x + 1 \quad \text{for all } x, y \in [0,1]$$

  is called the *probabilistic S-implication* based on the copula C,
- the function $\tilde{I}^*_C : [0,1]^2 \to [0,1]$ given by

$$\tilde{I}^*_C(x,y) = y + C(1-x, 1-y) \quad \text{for all } x, y \in [0,1]$$

  is called the *survival S-implication* based on the copula C.

Strategy 1: Combinations of fuzzy logical operators

Classes related to probability theory

These classes of fuzzy implications take into consideration both imprecision modelled by fuzzy concepts and randomness described by tools originated by probability theory.

Strategy 2: Use of unary monotone functions

Yager's f and g-generated implications

Definition (Yager, 2004)

Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing and continuous function with $f(1) = 0$. The function $I_f : [0,1]^2 \to [0,1]$ defined by

$$I_f(x,y) = f^{-1}(x \cdot f(y)), \quad x, y \in [0,1]$$

with the understanding $0 \cdot \infty = 0$, is called an *f-generated implication*.

Strategy 2: Use of unary monotone functions

Yager's f and g-generated implications

Definition (Yager, 2004)

Let $g : [0,1] \to [0,\infty]$ be a strictly increasing and continuous function with $g(0) = 0$. The function $I_g : [0,1]^2 \to [0,1]$ defined by

$$I_g(x,y) = g^{(-1)}\left(\frac{1}{x} \cdot g(y)\right), \quad x,y \in [0,1]$$

with the understanding $\frac{1}{0} = \infty$ and $\infty \cdot 0 = \infty$, is called a *g-generated implication*, where the function $g^{(-1)}$ is the pseudo-inverse of g.

Strategy 2: Use of unary monotone functions

Generalizations of Yager's f and g-generated implications

Several generalizations have been proposed by generalizing either the expression or the properties of f and g.

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Generalizations of Yager's f and g-generated implications

Several generalizations have been proposed by generalizing either the expression or the properties of f and g.

1. In f-generated implications, x is generalized to an increasing $g : [0,1] \to [0,1]$ such that $g(0) = 0$ and $g(1) = 1 \Rightarrow$ (f, g)-implications.

   *A. Xie, H. Liu: A generalization of Yager's f-generated implications. Int. Journal of Approximate Reasoning **54**(1): 35-46 (2013).*

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2. In g-generated implications, the product is generalized to an increasing operator $u_g : [1, +\infty] \to [0, +\infty]$ such that $u_g(+\infty, 0) = +\infty$ and $u_g(1, y) = y$ for all $y \in [0, g(1)]$.

   *F.-X. Zhang, H.-W. Liu: On a new class of implications: -implications and the distributive equations. Int. Journal of Approximate Reasoning, Vol. **54**, Issue 8, pp. 1049-1065, 2013.*

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   F.-X. Zhang, H.-W. Liu: On a new class of implications: -implications and the distributive equations. Int. Journal of Approximate Reasoning, Vol. ***54****, Issue 8, pp. 1049-1065, 2013.*

3. Changing x and $\frac{1}{x}$ by $\frac{x}{e}$ and $\frac{e}{x}$ with $e \in (0, 1]$.

   R. Fernandez-Peralta, S. Massanet: On the Characterization of a Family of Generalized Yager's Implications. IPMU (1) 2018: 636-648.

Strategy 2: Use of unary monotone functions

Other classes related to Yager's classes

Definition (Massanet, Torrens, 2011)

Fix an $e \in (0,1)$ and let $h : [0,1] \to [-\infty, \infty]$ be a strictly increasing and continuous function with $h(0) = -\infty$, $h(e) = 0$ and $h(1) = +\infty$. The function $I^h : [0,1]^2 \to [0,1]$ defined by

$$I(x,y) = \begin{cases} 1 & \text{if } x = 0, \\ h^{-1}(x \cdot h(y)) & \text{if } x > 0 \text{ and } y \leq e, \\ h^{-1}\left(\frac{1}{x} \cdot h(y)\right) & \text{if } x > 0 \text{ and } y > e, \end{cases}$$

is called an h-implication.

Strategy 2: Use of unary monotone functions

Generalizations of h-implications

Several generalizations have been proposed:

Strategy 2: Use of unary monotone functions

Generalizations of *h*-implications

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   S. Massanet, J. Torrens: On a new class of fuzzy implications: h-Implications and generalizations. Inf. Sci. ***181****(11): 2111-2127 (2011).*

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   S. Massanet, J. Torrens: On a new class of fuzzy implications: h-Implications and generalizations. Inf. Sci. ***181****(11): 2111-2127 (2011).*

3. Both generalizations together.

   S. Massanet, J. Torrens: On a new class of fuzzy implications: h-Implications and generalizations. Inf. Sci. ***181****(11): 2111-2127 (2011).*

Strategy 2: Use of unary monotone functions

Generalizations of h-implications

4. Changing the inner products by a minimum and a maximum $\Rightarrow$ (h, min)-implications.

   *H.-W. Liu : A new class of fuzzy implications derived from generalized h-generators. Fuzzy Sets and Systems **224**(11): 63-92 (2013).*

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5. Many others...

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   H.-W. Liu : A new class of fuzzy implications derived from generalized h-generators. Fuzzy Sets and Systems ***224****(11): 63-92 (2013).*

5. Many others...

Check the excellent review on this topic:

D. Hlinená, M. Kalina, P. Král': Implication Functions Generated Using Functions of One Variable. In Advances in Fuzzy Implication Functions, pp. 125-153 (2013).

Strategy 3: Constructions from other implications

Given one or several fuzzy implication functions, there exist many construction methods of new fuzzy implication functions. Let I, J be fuzzy implication functions and N a fuzzy negation:

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$$I_N(x,y) = I(N(y), N(x)), \quad x, y \in [0,1].$$

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2. The medium contrapositivisation

$$I_N^m(x,y) = \min\{I(x,y) \vee N(x), I_N(x,y) \vee y\}, \quad x, y \in [0,1].$$

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3. The φ-conjugation

$$I_\varphi(x,y) = \varphi^{-1}(I(\varphi(x), \varphi(y))), \quad x, y \in [0,1],$$

where φ is an automorphism on $[0,1]$.

Strategy 3: Constructions from other implications

Let I, J be fuzzy implication functions and N a fuzzy negation:

4. The $\min$ and $\max$ operations:

$$(I \vee J)(x,y) = \max\{I(x,y), J(x,y)\}, \quad x, y \in [0,1],$$

$$(I \wedge J)(x,y) = \min\{I(x,y), J(x,y)\}, \quad x, y \in [0,1].$$

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5. The convex combination with $\lambda \in [0,1]$:

$$I_{I,J}^{\lambda}(x,y) = \lambda \cdot I(x,y) + (1-\lambda) \cdot J(x,y), \quad x, y \in [0,1].$$

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6. The $\circledast$-method:

$$(I \circledast J)(x,y) = I(x, J(x,y)), \quad x, y \in [0,1].$$

*N.R. Vemuri, B. Jayaram: Representations through a monoid on the set of fuzzy implications, Fuzzy Sets and Systems, **247** (2014) 51-67.*

Strategy 3: Constructions from other implications

Let I, J be fuzzy implication functions and N a fuzzy negation:

7. The aggregation method with an aggregation function F

$$\mathcal{I}(x,y) = F(I(x,y), J(x,y)), \quad x, y \in [0,1].$$

T. Calvo, J. Martín, G. Mayor: Aggregation of implication functions. In Proceedings of EUSFLAT-13, pp. 569-574. Atlantis Press, 2013.

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8. The *FNI*-method with an aggregation function F such that $F(0,1) = 1$:

$$I_{F,N,I}(x,y) = F(N(x), I(x,y)), \quad x,y \in [0,1].$$

*I. Aguiló, J. Suñer, J. Torrens: How to modify a fuzzy implication function to satisfy a desired property, International Journal of Approximate Reasoning **103** (2018) 168-183.*

Strategy 3: Constructions from other implications

Let I, J be fuzzy implication functions and N a fuzzy negation:

7. Construction based on semi-copulas B:

$$J_{I,B}(x,y) = I(x, B(x,y)), \quad x, y \in [0,1].$$

M. Baczyński, P. Grzegorzewski, R. Mesiar, P. Helbin, W. Niemyska: Fuzzy implications based on semicopulas. Fuzzy Sets and Systems ***323****: 138-151 (2017)*

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8. The quadratic polynomial construction method:

$$I_F(x,y) = F(x,y,I(x,y)), \quad x, y \in [0,1]$$

where $F : [0,1]^3 \to [0,1]$ is a polynomial quadratic function.

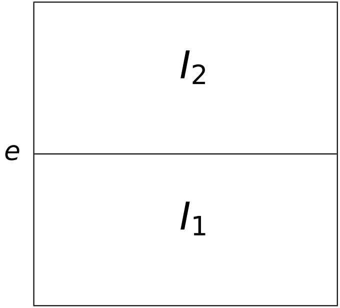
A. Kolesárová, S. Massanet, R. Mesiar, J.V. Riera, J. Torrens: Polynomial constructions of fuzzy implication functions: The quadratic case. Inf. Sci. ***494****: 60-79 (2019)*

Strategy 3: Constructions from other implications

Threshold constructions and ordinal sums

Let I_1, I_2,..., I_n be fuzzy implication functions:

1. The horizontal threshold method with $e \in [0, 1]$

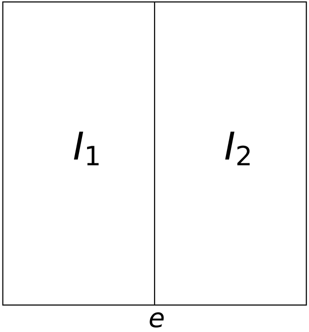


S. Massanet, J. Torrens: Threshold generation method of construction of a new implication from two given ones. Fuzzy Sets and Systems ***205****: 50-75 (2012)*

Strategy 3: Constructions from other implications

Threshold constructions and ordinal sums

2. The vertical threshold method with $e \in [0, 1]$

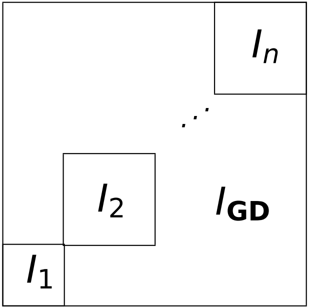


*S. Massanet, J. Torrens: On the vertical threshold generation method of fuzzy implication and its properties. Fuzzy Sets and Systems **226**: 32-52 (2013)*

Strategy 3: Constructions from other implications

Threshold constructions and ordinal sums

3. Ordinal sum “t-norm style”.

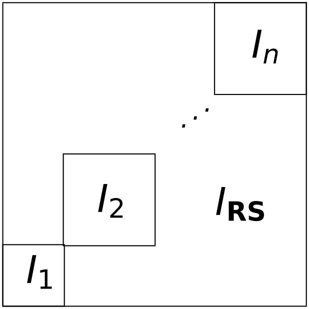


*Y. Su, A. Xie, H. Liu: On ordinal sum implications, Information Sciences, **293**, 251-262 (2015).*

Strategy 3: Constructions from other implications

Threshold constructions and ordinal sums

4. Other versions of the ordinal sum "t-norm style".



P. Drygaś and Anna Król: Various Kinds of Ordinal Sums of Fuzzy Implications, In: Novel Developments in Uncertainty Representation and Processing, 37-49. Springer (2016).

Strategy 3: Constructions from other implications

Threshold constructions and ordinal sums

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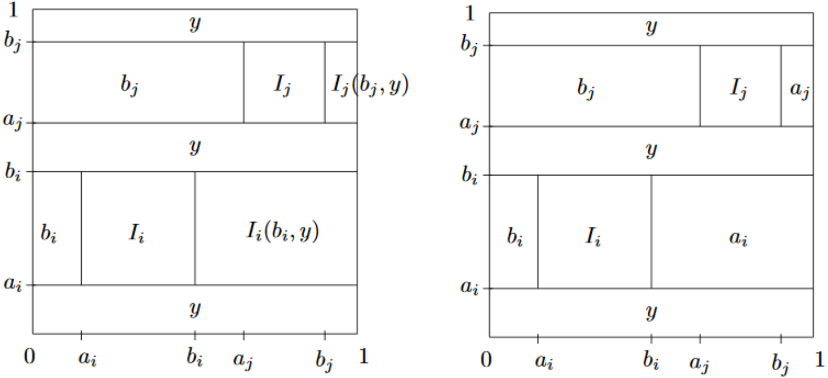
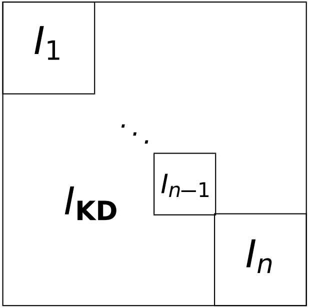


M. Baczyński, P. Drygas, A. Król, R. Mesiar: New types of ordinal sum of fuzzy implications. Proc. FUZZ-IEEE 2017: 1-6.

Strategy 3: Constructions from other implications

Threshold constructions and ordinal sums

6. Ordinal sum "(S, N)-style".



S. Massanet, J.V. Riera, J. Torrens: A New Look on the Ordinal Sum of Fuzzy Implication Functions. IPMU (1) 2016: 399-410.

New classes of fuzzy implication functions

Time for tidying up

Reasons for tidying up

There are too many classes of fuzzy implication functions introduced:

- without a proper motivation,
- without an in-depth study,
- without an axiomatic characterization.

leading to

- uselessness research,
- duplicated efforts.

Reasons for tidying up

Let us focus on these two papers:

M. Baczyński, P. Grzegorzewski, P. Helbin and W. Niemyska. Properties of the probabilistic implications and S-implications. Inf. Sci., ***331****:2-14, 2016.*

P. Helbin and M. Baczyński. Properties of the survival implications and S-implications. In J. M. Alonso et al., editors, Proc. of IFSA-EUSFLAT-15, pages 807-814. Atlantis Press, 2015.

In these papers, the authors study the fulfillment of the law of importation

$$I(T(x,y),z) = I(x, I(y,z)), \quad \text{for all} \quad x,y,z \in [0,1]. \qquad (\mathbf{LI_T})$$

and the Modus Ponens property

$$T(x, I(x,y)) \leq y, \quad x,y \in [0,1]. \qquad (\mathbf{MP}(\mathbf{T}))$$

where T is a t-norm for probabilistic and survival implications.

Reasons for tidying up

Definition

Let C be a copula. Then:

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$$I_C(x,y) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{C(x,y)}{x} & \text{if } x > 0, \end{cases}$$
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- the function $I_C^* : [0,1]^2 \to [0,1]$ given by
$$I_C^*(x,y) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{x+y-1+C(1-x,1-y)}{x} & \text{if } x > 0, \end{cases}$$
is called the *survival implication* based on the copula C.

They seem different classes. . .

Reasons for tidying up

but actually, they coincide!

> *S. Massanet, A. Pradera, D. Ruiz-Aguilera, J. Torrens: Equivalence and characterization of probabilistic and survival implications. Fuzzy Sets and Systems 359: 63-79 (2019)*

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Thus, unintentionally, the same results have been rediscovered in two different papers!

Reasons for tidying up

Theorem

Let $I : [0,1]^2 \to [0,1]$ be a binary function. The following statements are equivalent:

i) *I is a probabilistic implication derived from a copula C.*

ii) *I is a survival implication derived from a copula C'.*

iii) *I satisfies* **(I1)**, *$I(1,y) = y$, $I(0,y) = 1$ for all $y \in [0,1]$, the property*

$$x_2 I(x_2, y_1) + x_1 I(x_1, y_2) \leq x_1 I(x_1, y_1) + x_2 I(x_2, y_2)$$

for all $x_1 \leq x_2$ and $y_1 \leq y_2$, and

$$N_I(x) = N_{D_1}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, C and C' are uniquely given by

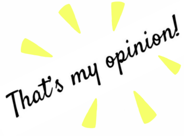
$$\begin{aligned} C(x,y) &= xI(x,y), \\ C'(x,y) &= x + y - 1 + (1-x)I(1-x, 1-y), \end{aligned}$$

for all $x, y \in [0,1]$.

Good and not-so-good practices

Good and not-so-good practices

It is time to analyze some practices on the study of classes of fuzzy implication functions.



Not-so-good practices

Ultimate goal: another class!

The following schema is recurrent in many papers:

Not-so-good practices

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Not-so-good practices

Ultimate goal: another class!

The following schema is recurrent in many papers:

1. A new class of fuzzy implication functions is proposed.
2. Some examples are given.
3. Some additional properties are studied for this family, many times in a lightly way.
4. Some intersections with the most important families are studied, many times in a lightly way.

Not-so-good practices

Ultimate goal: another class!

Motivation: A lot of classes of fuzzy implication functions are needed because in

> *E. Trillas, M. Mas, M. Monserrat, J. Torrens: On the representation of fuzzy rules. Int. J. Approx. Reasoning **48**(2): 583-597 (2008)*

it is said that for each application, depending on the context and the proper behaviour of the fuzzy "IF-THEN" rule, different implications can be suitable in each case.

Not-so-good practices

Ultimate goal: another class!

Why is a not-so-good practice?

Not-so-good practices

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Why is a not-so-good practice?

1. Too many classes are already available.

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Not-so-good practices

Ultimate goal: another class!

Why is a not-so-good practice?

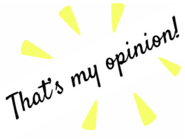
1. Too many classes are already available.
2. No application is given where the new class perform better than the existing classes.
3. The new class does not have any interesting feature in terms of the additional properties it satisfies.

Not-so-good practices

Ultimate goal: another class!

Why is a not-so-good practice?

1. Too many classes are already available.
2. No application is given where the new class perform better than the existing classes.
3. The new class does not have any interesting feature in terms of the additional properties it satisfies.
4. No characterization is usually proved.



Good practices

First desirable property, then new class

A different schema is the following one:

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A different schema is the following one:

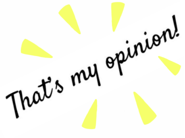
1. A desirable property (or a set of properties) is fixed.
2. The existing classes do not fulfill this property in general.
3. A new class that fulfills this property is proposed.
4. Other additional properties are analyzed.
5. The characterization of the new class is given.

Good practices

First desirable property, then new class

Why is a good practice?

1. A new class with a different behaviour with respect to the existing ones is presented.
2. The desirable property is connected with some applications.
3. If the characterization is achieved, the intersections with other classes are straightforward to obtain.



Good practices

First desirable property, then new class: an example

In 1982, Mizumoto and Zimmerman introduced the following example:

If the tomato is red, then it is ripe.
If the tomato is very red, then it is very ripe.
If the tomato is little red, then it is little ripe.



Good practices

First desirable property, then new class: an example

These fuzzy conditionals involve linguistic modifiers such as *very* or *little* which are usually modeled through Zadeh's potential modifiers:

- *very* x is computed as x^2,
- *little* x is computed as $x^{\frac{1}{2}}$.

Although Zadeh used the product t-norm $T_{\mathbf{P}}(x, y) = xy$, any continuous t-norm can be considered to model them.

Good practices

First desirable property, then new class: an example

It is common sense to expect that a proper fuzzy implication function when applied to these fuzzy conditionals

If the tomato is red, then it is ripe.
If the tomato is very red, then it is very ripe.
If the tomato is little red, then it is little ripe.

the same truth value is obtained. Indeed, whenever the same linguistic modifier is applied to both the antecedent and the consequent, the truth value of the fuzzy conditional remains the same.

Good practices

First desirable property, then new class: an example

From a continuous t-norm T, its powers can be defined. For all $x \in [0,1]$:

- $n \in \mathbb{Z}^+$, $n \geq 2$: $x_T^{(n)} = T(\overbrace{x, x, \ldots, x}^{n \text{ times}})$.
- $q \in \mathbb{Q}^+$: $x_T^{\left(\frac{1}{n}\right)} = \sup\{z \in [0,1] \mid z^{(n)} \leq x\}$, $\quad x_T^{\left(\frac{m}{n}\right)} = \left(x_T^{\left(\frac{1}{n}\right)}\right)^{(m)}$
- $r \in \mathbb{R}^+$: $x_T^{(r)} = \lim_{n \to \infty} x_T^{(a_n)}$ where $\lim_{n \to \infty} a_n = r$ with $a_n \in \mathbb{Q}^+$.

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- $r \in \mathbb{R}^+$: $\quad x_T^{(r)} = \lim_{n\to\infty} x_T^{(a_n)}$ where $\lim_{n\to\infty} a_n = r$ with $a_n \in \mathbb{Q}^+$.

Proposition

Let T be a continuous Archimedean t-norm with additive generator t. Then

$$x_T^{(r)} = t^{-1}(\min\{t(0), rt(x)\}) \text{ for all } x \in [0,1] \text{ and } r \in [0,+\infty].$$

Good practices

First desirable property, then new class: an example

Definition

Let $I : [0,1]^2 \to [0,1]$ be a binary function. It is said that I is *invariant with respect to T-powers*, or simply that it is *T-power invariant* when

$$I(x,y) = I\left(x_T^{(r)}, y_T^{(r)}\right). \qquad (\mathbf{PI_T})$$

holds for all real number $r > 0$ and for all $x, y \in [0,1]$ such that $x_T^{(r)}, y_T^{(r)} \neq 0, 1$.

Good practices

First desirable property, then new class: an example

Consider the fuzzy conditional

If the price of a computer is 450, then it is very good.

Most probably, we disagree with the previous statement and we will linguistically modify the consequent to fit it better to our idea.

Thus, we set the value of $x \to y$ as the highest possible power of y such that y up to this power becomes greater than or equal to x. Formally,

Good practices

First desirable property, then new class: an example

Definition

A binary operator $I : [0,1]^2 \to [0,1]$ is said to be a *T-power based implication* if there exists a continuous t-norm T such that

$$I(x,y) = \sup\{r \in [0,1] \mid y_T^{(r)} \geq x\} \qquad \text{for all} \;\; x, y \in [0,1].$$

If I is a T-power based implication, then it will be denoted by I^T.

Good practices

First desirable property, then new class: an example

Proposition

Let T be a continuous t-norm and I^T its power based implication. Then I^T is T-power invariant if and only if T is a strict ordinal sum t-norm, i.e., T is an ordinal sum t-norm of the form $T = \big(\langle a_j, b_j, T_j\rangle\big)_{j\in J}$ being T_j a strict t-norm for all summands $[a_j, b_j]$ such that $a_j \neq 0$.

Good practices

First desirable property, then new class: an example

Proposition

Let $I : [0,1]^2 \to [0,1]$ be a binary function. Then I is a T-power based implication for some nilpotent Archimedean t-norm T if and only if the following properties hold:

a) *I satisfies $I(x,y) = 1 \Leftrightarrow x \leq y$,*
b) *$I(x,y) \cdot I(y,0) = I(x,0)$ for all $x > y$,*
c) *the horizontal section $I(-,0) : [0,1] \to [0,1]$ is a strictly decreasing and continuous function with $I(1,0) = 0$.*

Moreover, in this case the t-norm T is the nilpotent Archimedean t-norm with additive generator $t(x) = I(x,0)$ for all $x \in [0,1]$.

Good practices

First desirable property, then new class: an example

For more details,

S. Massanet, J. Recasens, J. Torrens: Fuzzy implication functions based on powers of continuous t-norms. Int. J. Approx. Reasoning ***83****: 265-279 (2017)*

S. Massanet, J. Recasens, J. Torrens: Corrigendum to "Fuzzy implication functions based on powers of continuous t-norms" [Int. J. Approx. Reason. ***83*** *(2017) 265-279]. Int. J. Approx. Reasoning* ***104****: 144-147 (2019)*

S. Massanet, J. Recasens, J. Torrens: Some characterizations of T-power based implications. Fuzzy Sets and Systems ***359****: 42-62 (2019)*

Good practices

First desirable property, then new class: an example

Other examples of this practice:

*D. Paternain, H. Bustince, J. Fernández, J.A. Sanz, M. Baczyński, G. Beliakov, R. Mesiar: Strong Fuzzy Subsethood Measures and Strong Equalities Via Implication Functions. Multiple-Valued Logic and Soft Computing **22**(4-6): 347-371 (2014)*

*B. Jayaram, R. Mesiar: On special fuzzy implications. Fuzzy Sets and Systems **160**(14): 2063-2085 (2009)*

B. Jayaram: On the Law of Importation $(x \wedge y) \to z \equiv (x \to (y \to z))$ *in Fuzzy Logic. IEEE Trans. Fuzzy Systems **16**(1): 130-144 (2008)*

Good practices

Looking for Characterizations

An axiomatic characterization of a class of fuzzy implication functions is a set of additional properties which distinguish that class from the other classes.

- The *exchange principle*,

$$I(x, I(y, z)) = I(y, I(x, z)), \quad x, y, z \in [0, 1]. \qquad (\mathbf{EP})$$

Good practices

Looking for Characterizations

- The *exchange principle*,

$$I(x, I(y,z)) = I(y, I(x,z)), \quad x,y,z \in [0,1]. \qquad (\mathbf{EP})$$

- The *law of importation* with respect to a t-norm T,

$$I(T(x,y),z) = I(x, I(y,z)), \quad x,y,z \in [0,1]. \qquad (\mathbf{LI}(\mathbf{T}))$$

Good practices

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$$I(T(x, y), z) = I(x, I(y, z)), \quad x, y, z \in [0, 1]. \tag{\textbf{LI}(\textbf{T})}$$

- The *left neutrality principle*,

$$I(1, y) = y, \quad y \in [0, 1]. \tag{\textbf{NP}}$$

Good practices

Looking for Characterizations

- The *ordering property*,

$$x \leq y \Longleftrightarrow I(x, y) = 1, \quad x, y \in [0, 1]. \qquad (\mathbf{OP})$$

Good practices

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Good practices

Looking for Characterizations

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- The *identity principle*,

$$I(x,x) = 1, \quad x \in [0,1]. \tag{\textbf{IP}}$$

- The *contrapositive symmetry* with respect to a fuzzy negation N,

$$I(x,y) = I(N(y), N(x)), \quad x, y \in [0,1]. \tag{\textbf{CP}(\textbf{N})}$$

Good practices

Looking for Characterizations

Why is a good practice?

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1. Some insights on the possible applications are discovered.

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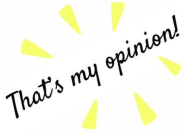
Good practices

Looking for Characterizations

Why is a good practice?

1. Some insights on the possible applications are discovered.
2. The intersections with other classes are easier to obtain.
3. The new class can be located in the set of all fuzzy implication functions.



Remember the definition of Yager's f-generated implications:

Definition

Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing and continuous function with $f(1) = 0$. The function $I_f : [0,1]^2 \to [0,1]$ defined by

$$I_f(x,y) = f^{-1}(x \cdot f(y)), \quad x, y \in [0,1]$$

with the understanding $0 \cdot \infty = 0$, is called an *f-generated implication*.

Good practices

Looking for Characterizations: an example

Theorem

Let $I : [0,1]^2 \to [0,1]$ be a binary function. Then the following statements are equivalent:

(i) *I is an f-generated implication with $f(0) < \infty$.*

(ii) *I satisfies **(LI)** with T_P and N_I is a strict negation.*

Moreover, in this case the f-generator is unique up to a positive multiplicative constant and it is given by $f(x) = N_I^{-1}(x)$.

Theorem

Let $I : [0,1]^2 \to [0,1]$ be a binary function. Then the following statements are equivalent:

(i) *I is an f-generated implication with $f(0) = \infty$.*

(ii) *I satisfies **(LI)** with T_P, I is continuous except at $(0,0)$ and $I(x,y) = 1 \Leftrightarrow x = 0$ or $y = 1$.*

Moreover, in this case the f-generator, that is unique up to a positive multiplicative constant, is given by

$$f(x) = \begin{cases} h_k^{-1}(x) & \text{if } k \leq x \leq 1, \\ \frac{1}{h_x^{-1}(k)} & \text{if } 0 < x < k, \\ \infty & \text{if } x = 0, \end{cases}$$

where $k \in (0,1)$.

Good practices

Looking for Characterizations: an example

These characterizations can be found in

S. Massanet, J. Torrens: On the characterization of Yager's implications. Inf. Sci. ***201****: 1-18 (2012).*

Another nice characterization (representation?) is available in terms of equivalence classes.

Definition

Let I and J be two fuzzy implication functions. If there exist some increasing automorphism $\varphi : [0,1] \to [0,1]$ such that

$$J(x,y) = \varphi(I(x, \varphi^{-1}(y))), \quad x, y \in [0,1]$$

we say that J is a φ-pseudo conjugate of I.

Definition

Let I be a fuzzy implication function. The equivalence class containing I can be given by

$$[I] = \{J \in \mathcal{FI} \mid J \text{ is a } \varphi - \text{pseudo conjugate of } I\}.$$

Denoting by $\mathbb{I}_{\mathbb{F},\infty}$ and $\mathbb{I}_{\mathbb{F},1}$ the classes of f-generated implications with $f(0) = +\infty$ and $f(0) < +\infty$,

Theorem

$$\mathbb{I}_{\mathbb{F},\infty} = [I_{\mathbf{YG}}].$$

Theorem

$$\mathbb{I}_{\mathbb{F},1} = [I_{\mathbf{RC}}].$$

More details in:

*N.R. Vemuri, B. Jayaram: Representations through a monoid on the set of fuzzy implications, Fuzzy Sets and Systems, **247** (2014) 51-67.*

Other recent characterizations:

I. Aguiló, J. Suñer, J. Torrens: A characterization of residual implications derived from left-continuous uninorms. Inf. Sci. ***180****(20): 3992-4005 (2010)*

Y. Shi, B. Van Gasse, D. Ruan, E.E. Kerre: On the first place antitonicity in QL.implications. Fuzzy Sets and Systems ***159****(22): 2988-3013 (2008)*

S. Massanet, A. Pradera, D. Ruiz-Aguilera, J. Torrens: From three to one: Equivalence and characterization of material implications derived from co-copulas, probabilistic S-implications and survival S-implications. Fuzzy Sets and Systems ***323****: 103-116 (2017)*

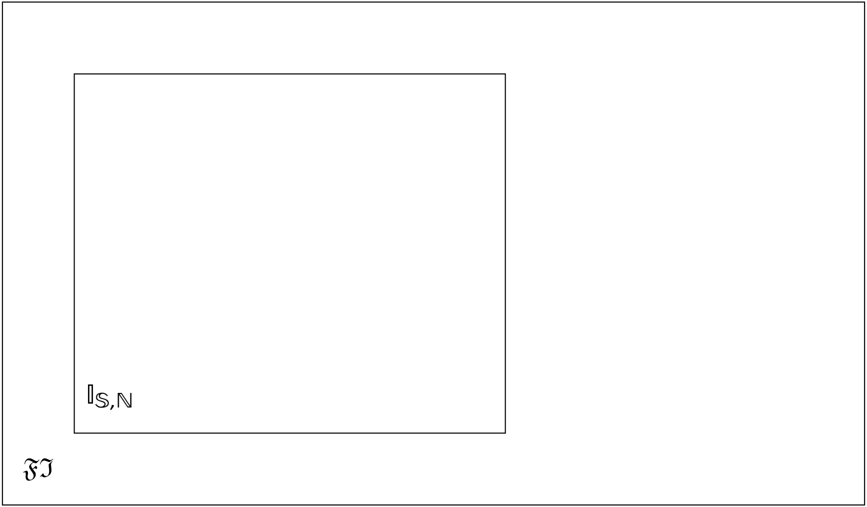
Good practices

Looking for Characterizations

$\mathfrak{F}\mathfrak{I}$

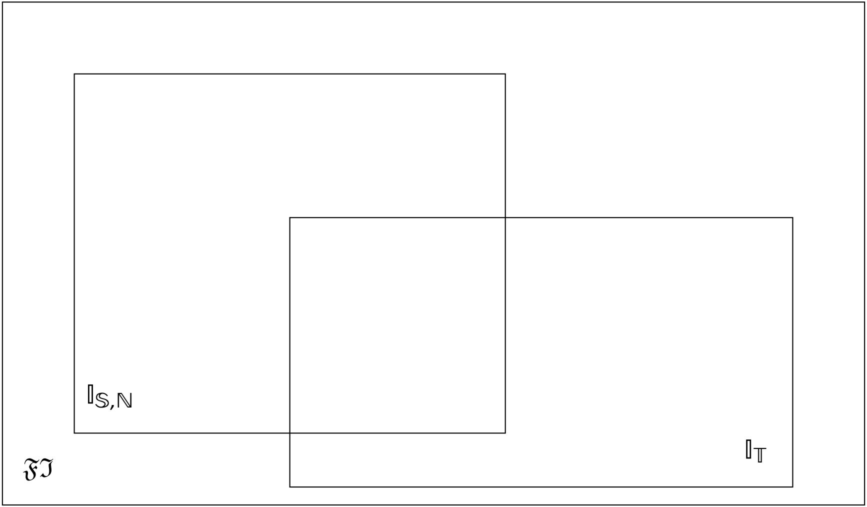
Good practices

Looking for Characterizations



Good practices

Looking for Characterizations



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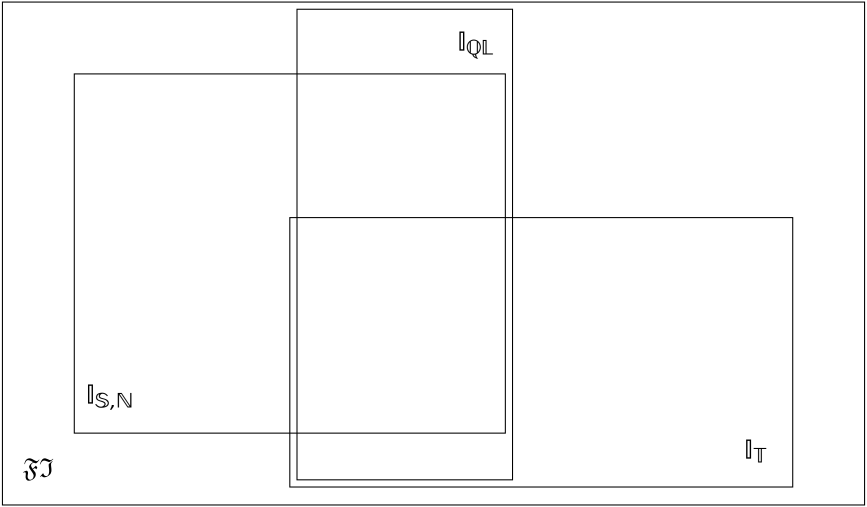

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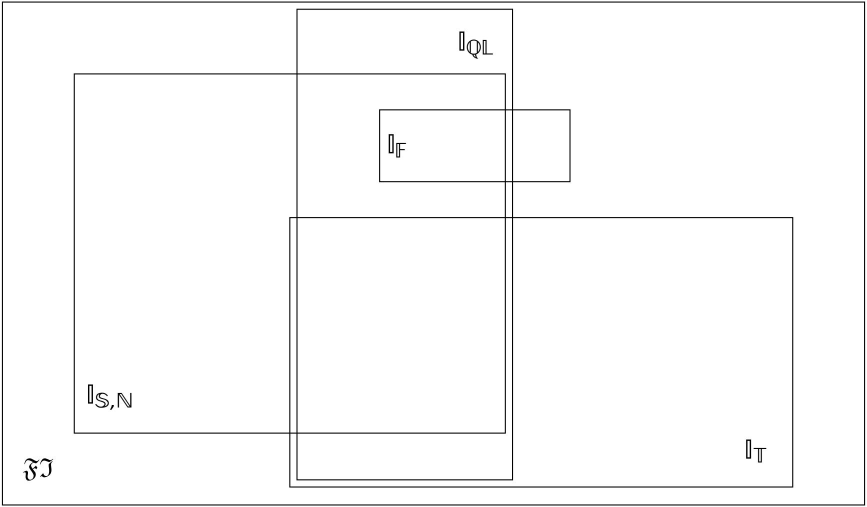

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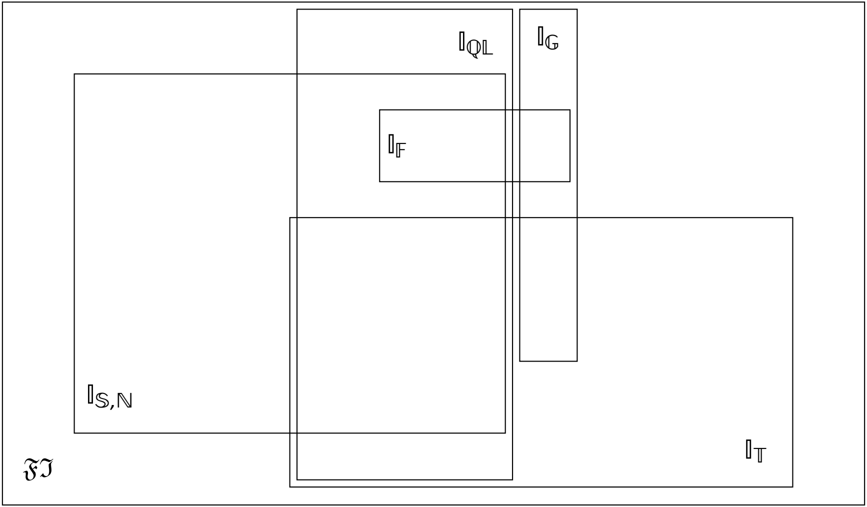

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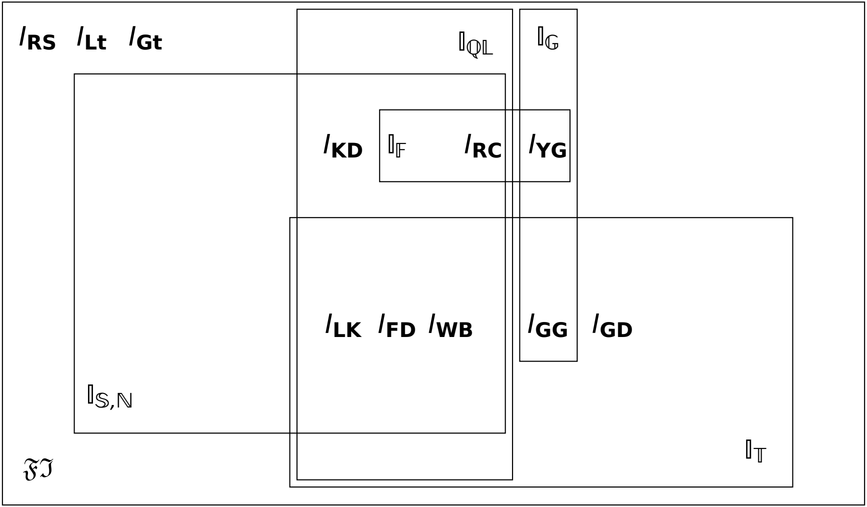

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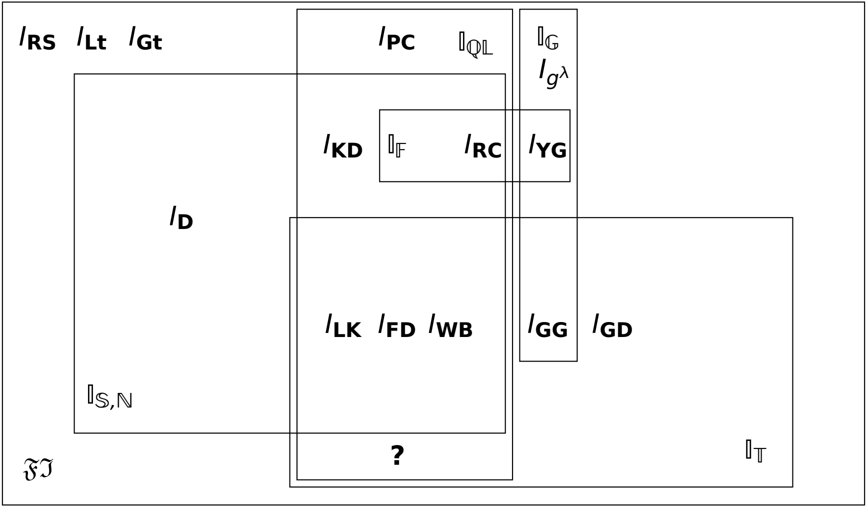

Conclusions and looking to the future

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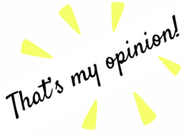

Thank you for your attention!